A STOCHASTIC MODEL FOR HIV TRANSMISSION UNDER PREVENTIVE STRATEGY USING EXPONENTIAL - GEOMETRIC DISTRIBUTION

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ABSTRACT

This paper focuses on the study of a stochastic model for estimating seroconversion time of HIV infected using preventive strategy. The concept of preventive strategy is gaining the greater importance now a days in view of both prevention and delaying the process of seroconversion. In this paper, the stochastic model for the estimation of expected time to seroconversion and variance are derived under the assumption that, threshold level of antigenic diversity is a random variable which follows exponential- geometric distribution. Numerical illustration is provided using simulated data.

Key words: Acquired Immuno Deficiency Syndrome, Antigenic Diversity Threshold, Human Immuno Deficiency Virus, Preventive Strategy, Seroconversion

INTRODUCTION

The science of epidemiology has great implication in the applications of epidemic disease Acquired Immuno Deficiency Syndrome (AIDS) is an infectious disease caused by retrovirus called Human Immuno Deficiency virus (HIV). The transmission of HIV from an infected partner to an uninfected person takes place mostly during sexual contacts. If more and more of HIV are getting transmitted from the infected person to the uninfected, the antigenic variation would be on the increase. This is due to the fact that the HIV is retrovirus and RNA based. The time to seroconversion from the point of infection depends upon what is known as antigenic diversity, which acts against the immune ability of an individual. Every individual has a threshold level of antigenic diversity. If the antigenic diversity due to acquiring more and more of HIV due to homo or hetro sexual contacts exceeds the threshold level, the immune system of human body is completely suppressed which is turn leads to seroconversion. For detailed study of antigenic diversity threshold and its estimation one can refer to Nowak and May (1991) and Stilianakis et al (1994).

An individual takes some precautionary measures with a view to avoid the HIV getting transmitted and this is called alertness. An effective and also a powerful methods or strategy against the HIV infection used to adopted preventive measures, so that the possibility of getting infected is completely ruled out. This
It is well known fact that the use of condoms during the sexual contacts is the most useful preventive strategy against HIV infection. The effectiveness of the adoption preventive strategies in the elongation of the expected time to seroconversion is studied by introducing a stochastic model.

A stochastic model for the HIV transmission under alertness, under the assumption that the threshold level of antigenic diversity is a random variable which follows a gamma distribution; mixed exponential distribution and exponentiated exponential distribution has been discussed by Kannan et al. (2007, 2011 and 2012). In this paper using the concept of alertness and preventive strategy the stochastic model for the statistical measures of time to seroconversion are derived under the assumption that the threshold level of antigenic diversity is a random variable which follows an exponential-geometric distribution. In developing such a stochastic model, the concept of shock model and cumulative damage process developed by Esary et al. (1973) is used. In this study the theoretical results are widely substantiated using numerical data simulated.

MODEL

This model portrays a situation, which is more realistic and has concurrence with the medical findings. A person may be alert in any contact with probability ‘p’ and inalert with probability ‘q’, so that p + q = 1. At the same time transmission of HIV during a contact is not a sure event according to medical findings. So it is assumed that the probability of transmission or infection during a contact occurs with probability δ < 1. Hence one can visualise the following situations.

i. A person is alert with probability ‘p’ and during that contact invasion does not take place.
ii. A person is inalert with probability ‘q’ and transmission occurs with probability δ.
iii. A person is inalert with probability ‘q’ and transmission does not occur and the probability of this event is (1 – δ).

ASSUMPTIONS OF THE MODEL

- Sexual contact is the only source of HIV transmission.
- During any contact in which a person is inalert the transmission of HIV is a sure event.
- A person is alert in a single contact with probability p, and inalert with probability q, so that p + q = 1.
- The damage caused to the immune system due to the antigenic diversity is linear and additive.
- The total damage exceeds a threshold level Y, which itself is a random variable, the seroconversion occurs and a person is recognized as infected.

NOTATIONS

\[ X_i : \text{A random variable denoting the increase in the antigenic diversity arising due to the HIV transmitted during the } i^{\text{th}} \text{ contact } X_1, X_2, \ldots, X_k \text{ are continuous i.i.d. random variables, with p.d.f. } g(.) \text{ and c.d.f. } G(.) \]
Y : A random variable representing antigenic diversity threshold and follows exponential-geometric distribution with parameters $\beta$ and $p$, the p.d.f. being $h(.)$ and c.d.f. $H(.)$.

$U_i$ : A continuous random variable denoting the inter-arrival times between successive contacts with p.d.f $f(.)$ and c.d.f. $F(.)$.

Z : The random variable representing the time between damages.

$g_k(.)$ : The p.d.f of random variable $\sum_{i=1}^{k} X_i$.

$F_k(.)$ : The $k^{th}$ convolution of $F(.)$.

T : A continuous random variable denoting the time to seroconversion with p.d.f. $l(.)$ and c.d.f. $L(.)$.

$V_k(.)$ : Probability of exactly $k$ contacts in $(0, t]$.

$l^*(s)$ : The Laplace transform of $l(t)$.

$f^*(s)$ : The Laplace transform of $f(t)$.

**RESULTS**

The probability density function of exponential-geometric distribution is,

$$h(y) = \beta (1 - p)e^{-\beta y}(1 - pe^{-\beta y})^2$$

and its distribution function is

$$H(y) = (1 - e^{-\beta y})(1 - pe^{-\beta y})^{-1}$$

And

$$\bar{H}(y) = \frac{e^{-\beta y}(1 - p)}{(1 - pe^{-\beta y})}$$

... (1)

Since Y is taken to be exponential-geometric distribution ($\beta$, $p$)

It can be shown that,

$$P\left[\sum_{i=1}^{k} X_i < Y\right] = \int_{0}^{\infty} g_k(x)\bar{H}(x) \, dx$$

... (2)
Where, \( H(x) = 1 - H(x) \)

Substituting (1) in (2), we get

\[
P \left[ \sum_{i=1}^{k} X_i < y \right] = \theta(1 - p) \sum_{i=1}^{k} \left[ \frac{p^{i-1}}{\theta + i\beta} \right]
\]

We define the survival function as \( S(t) \)

\[
S(t) = P(T > t) = \sum_{k=0}^{\infty} \Pr\{\text{there are exactly } k \text{ contacts in } (0,t]\}
\times \Pr\{\text{the cumulative total of antigenic diversity } < Y\}
\]

\[
S(t) = \sum_{k=0}^{\infty} V_k(t) P \left[ \sum_{i=1}^{k} X_i < y \right]
\]

\[
S(t) = \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] \theta(1 - p) \sum_{i=1}^{k} \left[ \frac{p^{i-1}}{\theta + i\beta} \right]
\]

\[
L(t) = 1 - S(t) = 1 - \left\{ \sum_{k=0}^{\infty} [F_k(t) - F_{k+1}(t)] \theta(1 - p) \sum_{i=1}^{k} \left[ \frac{p^{i-1}}{\theta + i\beta} \right] \right\}
\]

\[
l(t) = -\sum_{k=0}^{\infty} [f^*_k(t) - f^*_{k+1}(t)] \theta(1 - p) \sum_{i=1}^{k} \left[ \frac{p^{i-1}}{\theta + i\beta} \right]
\]

Taking Laplace transform of \( l(t) \) is,

\[
l'(s) = -\sum_{k=0}^{\infty} [f^*_k(s) - f^*_{k+1}(s)] \theta(1 - p) \sum_{i=1}^{k} \left[ \frac{p^{i-1}}{\theta + i\beta} \right] \quad \text{... (3)}
\]

Where \( f^*(s) \) is the Laplace transform of \( f(.) \).

But the c.d.f of \( Z \) is given by

\[
F(Z) = \delta q \sum_{n=0}^{\infty} [p + q (1 - \delta)]^n g_{n+1}(z)
\]

Taking Laplace Stieljies transform of \( F(Z) \), we get,
\[ F^*(s) = \int_0^\infty e^{-st} dF(Z) \]

\[ = \delta q G^*(s) \sum_{n=0}^\infty [(p + q(1 - \delta))G^*_s(s)]^n \]

Hence,

\[ F^*(s) = \frac{\delta q G^*(s)}{[1 - (p + q(1 - \delta))G^*(s)]} \]

The p.d.f of \( F^*(s) \) is,

\[ f^*(s) = \frac{\delta q g^*(s)}{\left[1 - (p + q(1 - \delta))g^*(s)\right]} \quad \text{... (4)} \]

\[ E(T) = -\frac{dI^*(s)}{ds} \bigg|_{s=0} \]

\[ = -\left\{ - \sum_{k=0}^\infty [f_k^*(s) - f_{k+1}^*(s)] \left[ \theta(1 - p) \sum_{i=1}^k \frac{p^{i-1}}{\theta + i\beta} \right] \right\} \]

\[ = \{(f^*(s))^k [1 - f^*(s)]\} \left[ \theta(1 - p) \sum_{i=1}^k \frac{p^{i-1}}{\theta + i\beta} \right] \quad \text{... (5)} \]

Substitute (4) in (5), we get,

\[ = \left\{ \left[\frac{\delta q g^*(s)}{[1 - (p + q(1 - \delta))g^*(s)]}\right]^k \left[1 - \frac{\delta q g^*(s)}{[1 - (p + q(1 - \delta))g^*(s)]}\right] \right\} \times \left[ \theta(1 - p) \sum_{i=1}^k \frac{p^{i-1}}{\theta + i\beta} \right] \]

\[ = \frac{\delta^k q^k [g^*(s)]^k - \delta^k q^k [g^*(s)]^{k+1}}{[1 - (p + q(1 - \delta))g^*(s)]^{k+1}} \times \left[ \theta(1 - p) \sum_{i=1}^k \frac{p^{i-1}}{\theta + i\beta} \right] \]

Assuming that \( g \sim \text{exp} (\mu) \), then,

\[ g^*(s) = \frac{\mu}{\mu + s} \]

\[ = \left[ \frac{\delta^k q^k \left[ \frac{\mu}{\mu + s} \right]^k - \delta^k q^k \left[ \frac{\mu}{\mu + s} \right]^{k+1}}{[1 - (p + q(1 - \delta))\frac{\mu}{\mu + s}]^{k+1}} \right] \times \left[ \theta(1 - p) \sum_{i=1}^k \frac{p^{i-1}}{\theta + i\beta} \right] \]
\begin{equation}
\frac{-d l^*(s)}{ds} = (\delta \mu')^k \left[ \frac{(s - \delta \mu')^{k+1} - s (k + 1)(s - \delta \mu')^{k+1}}{(s - \delta \mu')^{2k+2}} \right] \times \left[ \theta (1 - p) \sum_{i=1}^{k} \left[ \frac{p_r^{i-1}}{\theta + i \beta} \right] \right]\end{equation}

\begin{align}
E(T) &= \frac{-d l^*(s)}{ds} \bigg|_{s=0} = \frac{1}{\delta \mu'} \left[ \theta (1 - p) \sum_{i=1}^{k} \left[ \frac{p_r^{i-1}}{\theta + i \beta} \right] \right] \quad \ldots (6)
\end{align}

\begin{align}
E(T^2) &= \frac{d^2 l^*(s)}{ds^2} \bigg|_{s=0} \\
&= (\delta \mu')^k \left[ \frac{(s - \delta \mu')^{k+2}(-k)}{(s - \delta \mu')^{2k+4}} \right] \\
&\times \left[ \theta (1 - p) \sum_{i=1}^{k} \left[ \frac{p_r^{i-1}}{\theta + i \beta} \right] \right]^2 \\
E(T^2) &= \frac{d^2 l^*(s)}{ds^2} \bigg|_{s=0} = \frac{2}{\delta^2 q^2 \mu^2} \left[ \theta (1 - p) \sum_{i=1}^{k} \left[ \frac{p_r^{i-1}}{\theta + i \beta} \right] \right]^2 \quad \ldots (7)
\end{align}

\begin{equation}
V(T) = E(T^2) - [E(T)]^2 \quad \ldots (8)
\end{equation}

Substitute equation (6) and (7) in equation (8), we get,

\begin{align}
V(T) &= \frac{2}{\delta^2 q^2 \mu^2} \left[ \theta (1 - p) \sum_{i=1}^{k} \left[ \frac{p_r^{i-1}}{\theta + i \beta} \right] \right]^2 - \left[ \frac{1}{\delta \mu'} \left[ \theta (1 - p) \sum_{i=1}^{k} \left[ \frac{p_r^{i-1}}{\theta + i \beta} \right] \right] \right]^2 \\
&= \frac{1}{\delta^2 q^2 \mu^2} \left[ \theta (1 - p) \sum_{i=1}^{k} \left[ \frac{p_r^{i-1}}{\theta + i \beta} \right] \right]^2 \quad \ldots (9)
\end{align}

\begin{equation}
\text{(On simplification)}
\end{equation}

**SPECIAL CASES:**

When there is no alertness, then \(q=1, \delta' = 1\), so
\[ \mu_t = \frac{1}{\mu} \left[ \theta (1 - p) \sum_{i=1}^{k} \frac{p_i^{i-1}}{\theta + i\beta} \right] \]

In case of in-alertness,

\[ \mu_{ta} = \frac{1}{\delta q \mu} \left[ \theta (1 - p) \sum_{i=1}^{k} \frac{p_i^{i-1}}{\theta + i\beta} \right] \]

Therefore \( \mu_{ta} > \mu_t \) and this implies that mean time to seroconversion is larger in the case of alertness, which inversely proportional to the probability of non alertness ‘q’ which is an interesting result.

\[ \sigma_{ta}^2 = \frac{1}{\delta^2 q^2 \mu^2} \left[ \theta (1 - p) \sum_{i=1}^{k} \frac{p_i^{i-1}}{\theta + i\beta} \right]^2 \]

When \( q=1, \delta = 1 \), there is no alertness and variance in the case is given by ,

\[ \sigma_t^2 = \frac{1}{\mu^2} \left[ \theta (1 - p) \sum_{i=1}^{k} \frac{p_i^{i-1}}{\theta + i\beta} \right]^2 \]

Which are the results obtained by Kannan. and Chandrasekar.(2013).

**NUMERICAL ILLUSTRATIONS**

<table>
<thead>
<tr>
<th>( \mu )</th>
<th>( \theta = 0.2, p = 0.5, q = 0.3, \delta = 0.3, \beta = 0.4 )</th>
</tr>
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<tr>
<td></td>
<td>E(T)</td>
</tr>
<tr>
<td>1</td>
<td>1.587302</td>
</tr>
<tr>
<td>2</td>
<td>0.793651</td>
</tr>
<tr>
<td>3</td>
<td>0.529101</td>
</tr>
<tr>
<td>4</td>
<td>0.396825</td>
</tr>
<tr>
<td>5</td>
<td>0.317460</td>
</tr>
<tr>
<td>6</td>
<td>0.264550</td>
</tr>
<tr>
<td>7</td>
<td>0.226757</td>
</tr>
<tr>
<td>8</td>
<td>0.198413</td>
</tr>
<tr>
<td>9</td>
<td>0.176367</td>
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</tbody>
</table>
Figure -1

Table -2

<table>
<thead>
<tr>
<th>θ</th>
<th>$\beta = 0.4, q = 0.3,$</th>
<th>$\mu = 1, \delta = 0.5, \rho = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>E(T)</td>
<td>V(T)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.666667</td>
<td>0.4444444</td>
</tr>
<tr>
<td>0.2</td>
<td>1.111111</td>
<td>1.234568</td>
</tr>
<tr>
<td>0.3</td>
<td>1.428571</td>
<td>2.040816</td>
</tr>
<tr>
<td>0.4</td>
<td>1.666667</td>
<td>2.777778</td>
</tr>
<tr>
<td>0.5</td>
<td>1.851852</td>
<td>3.429355</td>
</tr>
<tr>
<td>0.6</td>
<td>2.000000</td>
<td>4.000000</td>
</tr>
<tr>
<td>0.7</td>
<td>2.121212</td>
<td>4.499541</td>
</tr>
<tr>
<td>0.8</td>
<td>2.222222</td>
<td>4.938272</td>
</tr>
<tr>
<td>0.9</td>
<td>2.307692</td>
<td>5.325444</td>
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</table>

Figure-2
Table 3

<table>
<thead>
<tr>
<th>p</th>
<th>$\theta = 0.2, q = 0.3, \mu = 1, \delta = 0.7, \beta = 0.4$</th>
</tr>
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<tr>
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<td>$E(T)$</td>
</tr>
<tr>
<td>0.1</td>
<td>0.38961</td>
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<tr>
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<td>0.34632</td>
</tr>
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<td>0.3</td>
<td>0.30303</td>
</tr>
<tr>
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<td>0.25974</td>
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<tr>
<td>0.5</td>
<td>0.21645</td>
</tr>
<tr>
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<td>0.17316</td>
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<td>0.7</td>
<td>0.12987</td>
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<td>0.08658</td>
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<tr>
<td>0.9</td>
<td>0.04329</td>
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</table>

Figure 3

Table 4

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\theta = 0.2, q = 0.3, \mu = 1, \delta = 0.3, \rho = 0.5$</th>
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<td></td>
<td>$E(T)$</td>
</tr>
<tr>
<td>0.1</td>
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<td>0.2</td>
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</tr>
<tr>
<td>0.3</td>
<td>2.222222</td>
</tr>
<tr>
<td>0.4</td>
<td>1.851852</td>
</tr>
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<td>1.587302</td>
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<td>0.7</td>
<td>1.234568</td>
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<td>0.8</td>
<td>1.111111</td>
</tr>
<tr>
<td>0.9</td>
<td>1.010101</td>
</tr>
</tbody>
</table>
\[ \begin{align*}
\theta &= 0.2, \rho = 0.5, \mu = 1, \beta = 0.4 \\
E(T) &\quad V(T) & E(T) & \quad V(T) & E(T) & \quad V(T) \\
\delta = 0.3 & \quad 0.1 & 2.380952 & 5.668934 & 1.428571 & 2.040816 & 1.020408 & 1.041233 \\
\delta = 0.5 & \quad 0.2 & 1.190476 & 1.417234 & 0.714286 & 0.510204 & 0.510204 & 0.260308 \\
\delta = 0.7 & \quad 0.3 & 0.793651 & 0.629882 & 0.476190 & 0.226757 & 0.340136 & 0.115693 \\
\delta = 0.9 & \quad 0.4 & 0.595238 & 0.354308 & 0.357143 & 0.127551 & 0.255102 & 0.065077 \\
\delta = 1.0 & \quad 0.5 & 0.476190 & 0.226757 & 0.285714 & 0.081633 & 0.204082 & 0.041649 \\
\delta = 1.1 & \quad 0.6 & 0.396825 & 0.157471 & 0.238095 & 0.056689 & 0.170068 & 0.028923 \\
\delta = 1.2 & \quad 0.7 & 0.340136 & 0.115693 & 0.204082 & 0.041649 & 0.145773 & 0.021250 \\
\delta = 1.3 & \quad 0.8 & 0.297619 & 0.088577 & 0.178571 & 0.031888 & 0.127551 & 0.016269 \\
\delta = 1.4 & \quad 0.9 & 0.264550 & 0.069987 & 0.158730 & 0.025195 & 0.113379 & 0.012855 \\
\end{align*} \]
CONCLUSIONS

From the table 1, the value of $\mu$ namely the parameter of the random variable denoting inter-arrival time between contacts increases then the expected time to seroconversion decreases and also variance time to seroconversion decreases.

As the value of $\theta$, which is namely, the parameter of the random variable $X_i$ denoting contribution to the antigenic diversity increases, then the expected time to seroconversion and variance time to
seroconversion both increases, as indicated in Table 2 and figure 2. This is due to the fact that \( g(.) \) is the distribution of \( X_i(j) \), the magnitude of contribution to antigenic diversity. Since \( 
abla(\Lambda) = \frac{1}{\theta} \), as \( \theta \) increases there is a decrease in the contribution of antigenic diversity.

From the Table 3, the behaviour of \( E(T) \) for fixed parameters when \( p \) is allowed to increase then mean time to seroconversion decreases. The same tendency is also noted on the variance of the seroconversion time of the HIV transmission.

It is observed from the Table 4 and also the graph as the value of \( \beta \) which is the parameter of the exponential-geometric distribution of the threshold increases the mean time to seroconversion decreases. It is also quite reasonable as regard the variation it could be seen that as the value of \( \beta \) increases, the variance decreases.

If \( q \) the probability of inalertness, the contribution to the antigenic diversity in successive contacts will be more and hence there is a decrease in the expected time to Seroconversion and also in its variance in Table 5, figure 5(a) and 5(b). It can easily observe that when the value of \( \delta \) increases, there is a decrease in the expected duration of Seroconversion irrespective of variation in \( q \) values. It is also quite reasonable as regards the variance for the fixed value.

REFERENCES