ISSN (Print): 2320-5504
ISSN (Online): 2347-4793

Asia Pacific Journal of Research
A peer reviewed international Journal
IMPACT FACTOR : 6.58
Print -ISSN -2320 - 5504
Online-E-ISSN - 2347 - 4793
University Grants Commission
Approved Journal S.N.45797

Recurrence Relation for Single & Product Moments of K^{th} Record Values From Muth Distribution and Its Characterization

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ABSTRACT

In this paper, we have derived some recurrence relations satisfied by single and product moments of k^{th} upper record values from Muth Distribution. Further a recurrence relation for single moment of the record values has been used for characterization of Muth distribution.

Keywords: *Single moments, product moments, k^{th} upper record values, recurrence relations, Muth Distribution, Characterization.*

1. Introduction

Chandler (1952) introduced the record values and discussed statistical properties of record values. He structured the theory of record values as a model for successive extremes in a sequence of independently and identically random variables. Record values are useful in many real life applications such as sports and athletic events, oil and mining surveys, climatology, medicine, traffic, industrial stress testing, bioscience and among others. Extreme values usually arises in lifetime data analysis e.g. when the system connected in series, after failing the first component, system breaks down and if the system connected in parallel form, during the last component fails, system stops working. Moreover, for large data, one might record the record values i.e. extremes. Let $\{X_n, n \ge 1\}$ be a sequence of independent and identically distributed (*iid*) random variable (*rv*) with cumulative distribution function *(cdf)* $F(x)$ and the probability density function (*pdf*) $f(x)$. Let $X_1, X_2, ..., X_n$ be a sample of size n from any population for any $k \in \square$, we define the sequence $\{U_n^{(k)}; n \ge 1\}$ of k^{th} upper record times of $\{X_n; n \ge 1\}$ as follows :

$$U_1^{(k)} = 1$$

$$U_{n+1}^{(k)} = \min\left\{ j > U_n^{(k)} : X_{j:j+k-1} > X_{U_n^{(k)}:U_{n+k-1}^{(k)}} \right\}$$

For k=1 and $n = 1, 2, ...,$ we can write $U_n^{(1)} = U_n$. $\{U_n, n \ge 1\}$ is the sequence of record times of $\{X_n, n \ge 1\}$.

The *pdf* of k^{th} upper record value $X_n^{(k)}; n \ge 1$ is given by Dziubdziela and Kopocinski (1976) as follows:

$$f_{X_n^{(k)}}(x) = \frac{k^n}{(n-1)!}\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x) \qquad -\infty < x < \infty \qquad (1.1)$$

The joint *pdf* of $X_m^{(k)} \,\&\, X_n^{(k)}, 1 \le m < n,\ n \ge 2,$ is given by Grudzien (1982) as follows:

ISSN (Print): 2320-5504
ISSN (Online): 2347-4793

$$f_{X_m^{(k)},X_n^{(k)}}(x,y)=\frac{k^n}{(m-1)!(n-m-1)!}\left[-\ln\overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)}\left[\ln\overline{F}(x)-\ln\overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}f(y),$$
$$-\infty<x<y<\infty \quad (1.2)$$

Where $\overline{F}(x)=1-F(x)$ denotes the survival function.

A random variable *X* is said to have a Muth distribution with shape parameter α if its *pdf* is given by

$$f(x)=\left(e^{\alpha x}-\alpha\right)\exp\left\{\alpha x-\frac{1}{\alpha}\left(e^{\alpha x}-1\right)\right\}, \qquad x>0,\alpha\in[0,1] \quad (1.3)$$

and corresponding *cdf* is given by

$$F(x)=1-\exp\left[\alpha x-\frac{1}{\alpha}\left(e^{\alpha x}-1\right)\right], \qquad x>0 \quad (1.4)$$

Therefore from (1.4) and (1.5), it is evident that the relation between *pdf* and *cdf* is given as

$$f(x)=\left(e^{\alpha x}-\alpha\right)\left(1-F(x)\right) \quad (1.5)$$

The relation in (1.5) will be utilized to derive the recurrence relation for moments of k^{th} upper record from Muth distribution. Muth distribution was introduced by Muth (1977) as an alternative reliability model. He also discussed its application in reliability theory. For next three decades, this distribution was unnoticed. Leemis and MacQueston (2008) summarized the interrelationship among several important univariate continuous distributions. Leemis and MacQueston (2008) has shown that Muth distribution reduces to standard exponential distribution if $\alpha\to 0$. Recently, Jodra *et. al* (2015) derived the various statistical properties of Muth distribution. Jodra *et. al* (2015) also obtained the maximum likelihood estimator, least square estimator, weighted least square estimator and moment estimator for shape parameter α and compared the efficiency of these estimator by simulation study.

Recurrence relation and characterization for k^{th} upper record values have been discussed by several authors. Dziubdziela and Kopociński (1976) have obtained the limiting properties of k^{th} record values. Glick (1978) has given the theory of breaking records and breaking boards. Grudzień (1982) has characterized distribution of time limits in record statistics as well as distributions and moments of linear records statistics from the sample of random numbers. Arnold *et. al* (1992) have discussed the concept of records. Grudzień and Szynal (1997) have discussed the characterization of continuous distributions via moments of k^{th} record values with random indices. Pawlas and Szynal (1999) have given the recurrence relations for single and product moments of k^{th} record values from Pareto, generalized Pareto and Burr distributions. Again Pawlas and Szynal (2000) have discussed the recurrence relations for single and product moments of k^{th} record values from Weibull distribution and a characterization. Khan *et. al* (2010) characterized the continuous distributions through record statistics. Khan *et. al* (2015) have deduced the relations for moments of k^{th} record values from exponented-Weibull lifetime distribution and a characterization. In this paper, we have established some recurrence relations for single & product moments of k^{th} upper record values from Muth distribution. Utilizing these results, we have also characterized Muth distribution based on upper record values.

This paper comprises four sections. In Section 2, we have established the recurrence relation based on single moment of k^{th} upper record values from Muth distribution. In Section 3, we have derived the recurrence relation by using product moment of k^{th} upper record values from Muth distribution. In section 4, we have characterized Muth distribution based on recurrence relation of single moment of k^{th} upper record values.

2. Relation for single moment from Muth distribution

In this section, we have derived the recurrence relation for single moment of k^{th} upper record values from Muth distribution.

Theorem 2.1. Fix a positive integer $k\geq 1$, for $n\geq 1$ and $r=0,1,2,\ldots$

$$\mu_{n:k}^{r}=\sum_{p=0}^{\infty}\frac{\alpha p}{p!}\frac{k}{r+p+1}\left[\mu_{n:k}^{r+p+1}-\mu_{n-1:k}^{r+p+1}\right]-\frac{\alpha k}{r+1}\left[\mu_{n:k}^{r+1}-\mu_{n-1:k}^{r+1}\right] \quad (2.1)$$

Proof: For $n\geq 1$ and $r=0,1,2,\ldots$ we have from (1.1) and (1.5)

ISSN (Print): 2320-5504
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$$\mu_{n:k}^{r} = E\left[\left(X_{U(n)}^{(k)}\right)\right]$$

$$= \int_0^\infty x^r \frac{k^n}{(n-1)!}\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx$$

Now using (1.5) we have,

$$= \frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}\left(e^{\alpha x}-\alpha\right)\overline{F}(x)dx$$

$$= \frac{k^n}{(n-1)!}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\int_0^\infty x^{r+p}\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k} dx - \frac{\alpha k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k} dx$$

Integrating the above equation by parts considering x^r for integration and rest part for differentiation, we have

$$\mu_{n:k}^{r} = \left[-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k^n}{(n-1)!}\int_0^\infty k\frac{x^{r+p+1}}{r+p+1}\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}\left(-f(x)\right)dx\right.$$

$$\left.-\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k^n}{(n-1)!}\frac{n-1}{r+p+1}\int_0^\infty x^{r+p+1}\left[-\ln \overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1} f(x)dx\right]$$

$$-\left[\frac{\alpha k^n k}{(n-1)!}\int_0^\infty \frac{x^{r+1}}{r+1}\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}\left(-f(x)\right)dx\right.$$

$$\left.+\alpha\frac{k^n}{(n-1)!}\frac{n-1}{r+1}\int_0^\infty x^{r+1}\left[-\ln \overline{F}(x)\right]^{n-2}\frac{-f(x)}{-\overline{F}(x)}\left(\overline{F}(x)\right)^{k} dx\right]$$

$$= \sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\frac{k}{r+p+1}\left[\frac{k^n}{(n-1)!}\int_0^\infty x^{r+p+1}\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx\right.$$

$$\left.-\frac{k^{n-1}}{(n-2)!}\int_0^\infty x^{r+p+1}\left[-\ln \overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1} f(x)dx\right]$$

$$-\frac{\alpha k}{r+1}\left[\frac{k^n}{(n-1)!}\int_0^\infty x^{r+1}\left[-\ln \overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx\right.$$

$$\left.-\frac{k^{n-1}}{(n-2)!}\int_0^\infty x^{r+1}\left[-\ln \overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1} f(x)dx\right]$$

$$\mu_{n:k}^{r} = \sum_{p=0}^{\infty}\frac{\alpha p}{p!}\frac{k}{r+p+1}\left[\mu_{n:k}^{r+p+1}-\mu_{n-1:k}^{r+p+1}\right]-\frac{\alpha k}{r+1}\left[\mu_{n:k}^{r+1}-\mu_{n-1:k}^{r+1}\right]$$

3. Relation for product moment from Muth distribution

In this section, we have derived the expression for recurrence relation for product moment based on k^{th} upper record values from Muth distribution.

Theorem 3.1. For $1 \le m \le n-1$ and $r, s = 0,1,2,.......$

ISSN (Print): 2320-5504
ISSN (Online): 2347-4793

$$\mu_{m,n:k}^{r,s} = \frac{k}{r+s+1}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\left[\mu_{m,n:k}^{r,s+p+1} - \mu_{m,n-1:k}^{r,s+p+1}\right] - \frac{k\alpha}{s+1}\left[\mu_{m,n:k}^{r,s+1} - \mu_{m,n-1:k}^{r,s+1}\right] \qquad (3.1)$$

and for $m \geq 1$ and $r,s = 0,1,2,...$

$$\mu_{m,m+1:k}^{r,s} = \frac{k}{r+s+1}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\left[\mu_{m,m+1:k}^{r,s+p+1} - \mu_{m:k}^{r,s+p+1}\right] - \frac{k\alpha}{s+1}\left[\mu_{m,n:k}^{r,s+1} - \mu_{m:k}^{r,s+1}\right] \qquad (3.2)$$

Proof: For $1 \leq m \leq n-1$ and $r,s = 0,1,2,...$ we have from (1.2) and (1.5)

$$\mu_{m,n:k}^{r,s} = E\left[\left(X_m^{(k)}\right)^r\left(X_n^{(k)}\right)^s\right]$$

$$= \frac{k^n}{(m-1)!(n-m-1)!}\int_0^{\infty} x^r\left[-\ln \overline{F}(x)\right]^{m-1}\frac{f(x)}{\overline{F}(x)}I(x)\,dx \qquad (3.3)$$

where

$$I(x) = \int_x^{\infty} y^s\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1} f(y)\,dy$$

$$I(x) = \int_x^{\infty} y^s\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1}\left(e^{-\alpha y} - \alpha\right)dy$$

Integrating the above integral by parts, we have

$$I(x) = -\frac{n-m-1}{s+p+1}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\int_x^{\infty} y^{s+p+1}\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1} f(y)\,dy$$

$$+\frac{k}{s+p+1}\sum_{p=0}^{\infty}\frac{\alpha^p}{p!}\int_x^{\infty} y^{s+p+1}\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1} f(y)\,dy$$

$$-\frac{\alpha(n-m-1)}{s+1}\int_x^{\infty} y^{s+1}\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-2}\left[\overline{F}(y)\right]^{k-1} f(y)\,dy$$

$$+-\frac{k\alpha}{s+1}\int_x^{\infty} y^{s+1}\left[\ln \overline{F}(x) - \ln \overline{F}(y)\right]^{n-m-1}\left[\overline{F}(y)\right]^{k-1} f(y)\,dy$$

Substituting this $I(x)$ in (3.3) and simplifying, we get the result.

4. Characterization of Muth Distribution

In this section, utilizing the result obtained in Section 2, we have characterized Muth distribution by using single moment of k^{th} upper record values. In characterization result, we try to find the distribution of random variables if certain statistical condition is fulfilled by random variables.

Theorem 3: Fix a positive $k \geq 1$ and let r be a non-negative integer. A necessary and sufficient condition for a random variable X to follow Muth distribution having *cdf* given in (1.2) is that

$$\mu_{n:k}^{r} = \sum_{p=0}^{\infty}\frac{\alpha p}{p!}\frac{k}{r+p+1}\mu_{n:k}^{r+p+1} - \sum_{p=0}^{\infty}\frac{\alpha p}{p!}\frac{k}{r+p+1}\mu_{n-1:k}^{r+p+1} - \frac{\alpha k}{r+1}\mu_{n:k}^{r+1} + \frac{\alpha k}{r+1}\mu_{n-1:k}^{r+1} \qquad (4.1)$$

ISSN (Print): 2320-5504
ISSN (Online): 2347-4793

Proof: The necessary part follows immediately form (2.1), on the other hand, if the recurrence relation (4.1) is satisfied then on rearranging the terms in (4.1) and using (1.1), we have

$$\frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx$$

$$=\sum_{p=0}^\infty \frac{\alpha^p}{p!}\frac{k}{r+p+1}\frac{k^n}{(n-1)!}\int_0^\infty x^{r+p+1}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx$$

$$-\sum_{p=0}^\infty \frac{\alpha^p}{p!}\frac{k}{r+p+1}\frac{k^{n-1}}{(n-2)!}\int_0^\infty x^{r+p+1}\left[-\ln\overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1} f(x)dx$$

$$-\frac{\alpha k}{r+1}\frac{k^n}{(n-1)!}\int_0^\infty x^{r+1}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx$$

$$+\frac{\alpha k}{r+1}\frac{k^{n-1}}{(n-2)!}\int_0^\infty x^{r+1}\left[-\ln\overline{F}(x)\right]^{n-2}\left(\overline{F}(x)\right)^{k-1} f(x)dx \qquad (4.2)$$

Integrating the first and third integral by parts of R.H.S. of (4.2)

Equation (4.2) becomes

$$\frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx$$

$$=\sum_{p=0}^\infty \frac{\alpha^p}{p!}k\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{r+p}\left[-\ln\left(1-F(x)\right)\right]^{n-1}\left(\overline{F}(x)\right)^{k} dx-\alpha k\frac{k^{n-1}}{(n-1)!}\int_0^\infty x^{r}\left[-\ln\left(1-F(x)\right)\right]^{n-1}\left(\overline{F}(x)\right)^{k} dx$$

$$\Rightarrow \frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1} f(x)dx$$

$$-\sum_{p=0}^\infty \frac{\alpha^p}{p!}\frac{k^n}{(n-1)!}\int_0^\infty x^{r+p}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k} dx+\alpha\frac{k^n}{(n-1)!}\int_0^\infty x^{r}\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k} dx=0$$

$$\frac{k^n}{(n-1)!}\int_0^\infty x^r\left[-\ln\overline{F}(x)\right]^{n-1}\left(\overline{F}(x)\right)^{k-1}\left[f(x)-\sum_{p=0}^\infty \frac{\alpha^p}{p!}x^p\overline{F}(x)+\alpha\overline{F}(x)\right]dx=0$$

Applying Müntz-Szász Theorem (see Hwang and Lin; 1984), we get

$$f(x)-\sum_{p=0}^\infty \frac{\alpha^p}{p!}x^p\overline{F}(x)+\alpha\overline{F}(x)=0$$

or,

$$f(x)=\left(e^{\alpha x}-\alpha\right)\left(1-F(x)\right)$$

Thus we get the relation given in (1.5).

References:

ISSN (Print): 2320-5504
ISSN (Online): 2347-4793

1. Arnold. B.C., Balakrishanan, N. and Nagaraja, H.N. (1992): Records, *John Wiley and Sons*, New York.
2. Chandler, K.M. (1952): The distribution and frequency of record values. *J. R. Statist. Soc., Ser. B*, **14**, 220-228.
3. Dziubdziela, W. and Kopociński, B. (1976): Limiting properties of k-th record values. *Appl. Math.*, **15**, 187-190.
4. Glick, N. (1978): Breaking records and breaking boards. *Am. Math*. Monthly, **85**, 2-26.
5. Grudzień, Z. (1982): Characterization of distribution of time limits in record statistics as well as distributions and moments of linear records statistics from the sample of random numbers. *Praca Doktorska, UMCS*, Lubin.
6. Grudzień, Z. and Szynal, D. (1997): Characterization of continuous distributions via moments of k-th record values with random indices. *J. Appl. Statist., Sci*., **5**, 259-266.
7. Hwang, J. S. and Lin, G.D. (1984): On a generalized moments problem II. *Proc. Amer. Math. Soc.*, **91**, 577-580.
8. Jodrá, P., Jimnénez-Gamero, M.D. and Alba-Fernández, M.V. (2015): On the Muth distribution. *Math. Model. Anal.*, **20(3)**, 291-310.
9. Khan, A.H., Faizan, M. and Haque, Z. (2010): Characterization of continuous distributions through record statistics. *Commun. Korean Math. Soc.,* **25**, 485-489.
10. Khan, R.U., Kulshrestha, A. and Khan, M.A. (2015): Relations for moments of k-th record values from exponential-Weibull lifetime distribution and a characterization. *Journal of the Egyptian Mathematical Society*, **23**, 558-562.
11. Leemis, L.M. and McQueston, J.T. (2008): Univariate distribution relationships. *Am. Stat.*, **62(1)**, 45-53.
12. Muth, E.J. (1977): Reliability models with positive memory derived from the mean residual life function. In. C.P. Tsokos and I. Shimi (Eds.), The Theory and Apllications of Reliability, *Academic Press, Inc.*, New York, **2**, 401-435.
13. Pawlas, P. and Szynal, D. (1999): Recurrence relations for single and product moments of k^{th} record values from Pareto, generalized Pareto and Burr distributions. *Comm. Statist. Theory. Methods*, **28**, 1699-1709.
14. Pawlas, P. and Szynal, D. (2000): Recurrence relations for single and product moments of k^{th} record values from Weibull distribution and a characterization. *J. Appl. Stat. Sci.*, **10**, 17-26.