A Sequential Test for the Mittag-Leffler Distribution

Babalola A. R. and T. O. Obilade

Department of Mathematics, Obafemi Awolowo University, Ile-Ife 220005, Nigeria

Abstract

We consider the classical Mittag-Leffler function under a sequential experimentation, upon the basis that it is a distribution and fits even better for a geometrically increasing sequence. It is no news that the Laplace Steiltjes transform of \( f(z) \) would yield an incomplete gamma \( s; \lambda, \alpha k \) distribution and itself generalizes more distribution than just few, when under certain constraint. We estimate the Operating Characteristics function of the sequential ratio test and made certain findings on the average amount of inspection required by the test.

Keywords: Operation Characteristics Function, Expected Average Sample Number, Sequential Probability Ratio Test (SPRT).

1. Introduction

The Mittag-Leffler function was introduced in 1903 in the context of summing a divergent series. It is of the form

\[
E_\alpha(z) = \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0
\]

(1)

Following from [2] let \( \Theta_{k=0}^\infty \) be a sequence of complex numbers satisfying the condition

\[
\lim \sup n \to \infty (|a_n|)^{\frac{1}{n}} = 1
\]

(2)

and \( f(z), g(z; \beta), h(z; \alpha) \) be the sums respectively of the first, second and third of the series on a disk \( D \), with \( D \{ z : z \in \mathbb{C}, |z| < 1 \} \) i.e.

\[
f(z) = \sum_{k=0}^{\infty} \theta_k \tilde{E}_k(z)
\]

(3)

\[
g(z; \beta) = \sum_{k=0}^{\infty} \theta_k E_{k,\beta}(z)
\]

(4)

the left hand side of (3) is seen as

\[
f(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0,
\]

(5)

with conditions as \( \{ \theta_k \} : \sum_{k=0}^{\infty} \theta_k = \lambda^{\alpha+1} \) and \( k \in \mathbb{Z}^+ \), the Laplace Steiltjes transform of \( f(z) \) would easily yield an incomplete gamma \( s; \lambda, \alpha k \) distribution, with some on \( k \), we have a complete gamma distribution.
2. The Notion of the Distribution

Provided that the experiment is less expensive, certain procedures are adopted; that is, if the sequential test is desired in a way that the probability error does exceed the error of rejecting a true null hypothesis \( \alpha \) and that of the accepting a false null hypothesis. Then we can conveniently carry out a sequential test with no conditions attached. Consider that

\[
\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \left( \frac{e^{(1+z)t} - e^t}{e^t - 1} \right)^N
\]

(6)

\[
\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \left( \frac{1}{N} \right)^N \frac{(e^{(1+z)t} - e^t)^N}{(e^t - 1)^N - n e^{(N+nx)t}}
\]

(7)

\[
= \sum_{m=0}^{\infty} B_m N \frac{t^{m-N}}{m!} \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \frac{e^{(N+nx)t}}{s!}
\]

(8)

\[
= \sum_{m=0}^{\infty} B_m N \frac{t^{m-N}}{m!} \sum_{n=0}^{N} \binom{N}{n} (-1)^{N-n} \frac{(N+nx)^s}{s!m!}
\]

(9)

\[
= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} B_m N \frac{t^{m-N}}{m!} \frac{(N+nx)^s}{s!m!} (10)
\]

with \( N = n \) conditioned as 1

\[
= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} B_m (1+x)^s \frac{t^{s+m-1}}{s!m!}
\]

(11)

\[
= \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} B_m \frac{[(1+x)t]^s}{s!} \frac{t^{m-1}}{m!}
\]

(12)

with \( m = 1 \)

\[
= \sum_{s=0}^{\infty} \frac{[(1+x)t]^s}{s!} \frac{t^m}{m!}
\]

(13)

\[
= \sum_{s=0}^{\infty} \frac{[(1+x)t]^s}{\Gamma(s+1)}
\]

(14)

compare with the right hand side of eqn (6), we have that \((1+x)t = z\). Hence we can easily say that

\[
\sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} = \sum_{s=0}^{\infty} \frac{[(1+x)t]^s}{\Gamma(s+1)} = \left( \frac{e^{(1+z)t} - e^t}{e^t - 1} \right)^N
\]

(15)

3. The Operating Characteristic Function of the Sequential Probability Ratio test

\( L(\alpha) \) is defined as the probability that the sequential process will terminate with accepting \( H_0 \) when \( \alpha \) is the true value of the parameter [Wald,1946]. Let \( E_\alpha(z) = f(z; \alpha) \) and then we say that the testing of \( L(z) \) for \( H_0 : \alpha = \alpha_0 \) against \( H_1 : \alpha = \alpha_1 \) for the Mittag-Leffler distribution is considered under the condition that

\[
E \left[ \frac{E_{\alpha_1}(z)}{E_{\alpha_0}(z)} \right]^h = 1
\]

(16)
This quantity is also equal to
\[
\int_{-\infty}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \left( \frac{z^k}{\Gamma(\alpha k + 1)} \right)^{\alpha_k + 1} dz = 1
\]  
(17)
\[
\frac{\Gamma(\alpha_0 k + 1)}{\Gamma(\alpha_1 k + 1)} \frac{1}{\Gamma(\alpha k + 1)} \int_{-\infty}^{\infty} z^k dz = 1
\]  
(18)
\[
h \ln \frac{\Gamma(\alpha_0 k + 1)}{\Gamma(\alpha_1 k + 1)} \left( \frac{2\alpha^{k+1}}{k+1} \right) + \ln c = 0
\]  
(19)
\[
h = 2 \ln \Gamma(\alpha k + 1) - 4 \frac{\alpha^{k+1}}{k+1}
\]  
(20)

at \( \alpha_0 = \alpha_1 \)

To find the likelihood of \( () \), we resolve to \( () \) which

\[
L(z; \alpha) = \prod_{i=1}^{m} \alpha_e = \prod_{i=1}^{m} \left( \frac{e^{(1+z)t} - e^t}{e^t - 1} \right)^N \]  
(21)

To find the SPRT of the function, we consider a hypothetical where a certain quantity \( k_0 \) is less than or equal to the valid function for the SPRT (say \( \lambda_m \)) and less than or equal to a certain quantity \( k_1 \) (as like \( k_0 \)), so that,

\[
k_0 \leq \left( \frac{e^{(1+\sum x)\frac{t_0}{t_1}} - e^{\frac{t_0}{t_1}}}{e^{\frac{t_0}{t_1}} - 1} \right) \leq k_1
\]  
(22)
\[
Nm \ln k_0 \leq \ln \left( \frac{e^{(1+\sum x)\frac{t_0}{t_1}} - e^{\frac{t_0}{t_1}}}{e^{\frac{t_0}{t_1}} - 1} \right)^N \leq Nm \ln k_1
\]  
(23)
\[
Nm \ln k_0 + \ln(e^{\frac{t_0}{t_1}} - 1) \leq \ln(e^{(1+\sum x)\frac{t_0}{t_1}} - e^{\frac{t_0}{t_1}}) \leq Nm \ln k_1 + \ln(e^{\frac{t_0}{t_1}} - 1)
\]  
(24)
\[
Nm \ln k_0 + \ln(e^{\frac{t_0}{t_1}} - 1) \leq \ln(e^{(1+\sum x)\frac{t_0}{t_1}} - e^{\frac{t_0}{t_1}}) \leq Nm \ln k_1 + \ln(e^{\frac{t_0}{t_1}} - 1)
\]  
(25)
\[
Nm \ln k_0 + \frac{t_0}{t_1} \leq (1 + \sum x) \frac{t_0}{t_1} - \frac{t_0}{t_1} \geq Nm \ln k_1 + \frac{t_0}{t_1}
\]  
(26)
\[
\frac{t_1}{t_0} (Nm \ln k_0) + 1 \leq (1 + \sum x) - 1 \leq \frac{t_0}{t_1} (Nm \ln k_1) + 1
\]  
(27)
\[
\frac{t_1}{t_0} (Nm \ln k_0) + 1 \geq \sum x \geq \frac{t_0}{t_1} (Nm \ln k_1) + 1
\]  
(28)

while we let \( C_0 \) be equal to \( \frac{t_0}{t_1} (Nm \ln k_0) + 1 \) and \( C_1 \) be \( \frac{t_0}{t_1} (Nm \ln k_1) + 1 \) so therefore that \( C_1 \leq \sum x \leq C_0 \) and we can in turn reject the null hypothesis \( H_0 : t = t_0 \) with the condition that \( C_1 \leq \sum x_i \) and accept \( H_1 \) in the decision otherwise we do not reject \( H_0 \).
4. The Average Amount of Inspection Required by the Test

We consider \( n \) as the number of observations required by the test and, \( E_\alpha(n) \) the expected value of \( n \) when \( \alpha \) is the true value of the parameter. In actual fact, \( E_\alpha(n) \) is called the average sample number function (ASN), [1] Hence the ASN for the Mittag-Leffler distribution is given by

\[
E_\alpha(z) = \left[ \ln \frac{\theta_1 z^k}{\Gamma(\alpha_1 k + 1)} \right] = \ln[z^k(\theta_1 - \theta_0) - \Gamma(\alpha_1 k + 1) - \Gamma(\alpha_0 k + 1)]
\]  

and, the expected value of the value \( z \) when \( \alpha \) is true is the true meaning of \( z \) and it is given as

\[
E_\alpha(n) = \frac{L(\alpha) \ln \frac{\beta}{1 - \alpha} + (1 - L(\alpha)) \ln \frac{1 - \beta}{\alpha}}{\ln[z^k(\alpha_1 - \alpha_0) - (\Gamma(\alpha_1 k + 1) - \Gamma(\alpha_0 k + 1))]}
\]

where \( z^k(\alpha_1 - \alpha_0) = \alpha \) and \( (\Gamma(\alpha_1 k + 1) - \Gamma(\alpha_0 k + 1)) \) is \( s \).

Furthermore, the first part of the numerator \( \ln \frac{\beta}{1 - \alpha} \) is \( h_1 \) and the second part \( \ln \frac{1 - \beta}{\alpha} \) is the \( h_0 \). The values of \( h_0 \) and \( h_1 \) are the intercepts and \( s \) is the slope of the line \( L_0 \) and \( L_1 \), whose form is found in [1].

References