Integral solutions of the ternary quadratic equation 5(x² + y²) - 9xy = (k² + 19)z²

R. Nandhini

Lecturer in Mathematics, Bharathidasan university Model College,
Thiruthuraipoondi-614806, Tamil Nadu, India.

ABSTRACT

The Ternary Quadratic Diophantine equation given by 5(x² + y²) - 9xy = (k² + 19)z² is analyzed for its different patterns of non-trivial integral solutions. A few interesting relations between the solutions and special polygonal numbers are exhibited.

Keywords: Ternary, Quadratic, Integral solution, special polygonal numbers, pentagonal numbers

Introduction:

The theory of Diophantine equations offers a rich variety of fascinating problems (1-5). For an extensive review of sizable literature and various problems, one may refer [6-20]. This communication concerns with yet another interesting ternary quadratic equation 5(x² + y²) - 9xy = (k² + 19)z² for determining its infinitely many non-trivial integral solutions. Also a few interesting relations among the solutions have been presented.

Notations Used:

- t_m^n- Polygonal number of rank n with size m.
- P_n^k- Pentagonal number of rank n with size k.
- S_n- Star number of rank n
- Pr_n- Pronic number of rank n
- Obl_n- Oblong number of rank n
- Sqp_n- Square pyramidal number of rank n

Method of Analysis:

The Ternary Quadratic Diophantine Equation to be solved for its non-zero distinct integral solution is

5(x² + y²) - 9xy = (k² + 19)z²

(1)

By assuming x = u + v, y = u - v

equation (1) reduces to u² - k²z² = 19(z² - v²)

(3)

PATTERN I

Equation (3) can be written as

\[
\frac{u - kz}{z - v} = \frac{19(z + v)}{u + kz} = \frac{p}{q}
\]

(4)

which is equivalent to the pair of equations
Using cross multiplication method, we get (5)

In view of (5), the solutions of (1) are

\[ x(p,q,k) = 19q^2(k-1)p^2+38pq(k+19) \]
\[ y(p,q,k) = 19q^2(k-1)p^2+2pq(k-19) \]
\[ z(p,q,k) = 19q^2+p^2 \]

A few interesting properties observed are as follows:

1. \( x(p,2,1) - y(p,2,1) - t_{19,p} + t_{6,p} \equiv 0 \) (mod 8)
2. \( x(2,q,1) + z(2,q,1) - t_{19,q} + t_{8,q} \equiv 4 \) (mod 129)
3. \( 2y(2,q,3) - x(2,q,3) - t_{19,q} + t_{4,q} \equiv -24 \) (mod 154)
4. \( y(1,q,2) + x(1,q,2) - S_t - t_{12,q} + t_{12,q} \equiv -5 \) (mod 152)
5. \( x(p,2,1) + 2y(p,2,1) - Obl_{p} + t_{10,p} \equiv 72 \) (mod 80)
6. \( x(1,q,1) + 2y(1,q,1) - t_{162,p} + t_{10,q} \equiv -4 \) (mod 188)

**PATTERN II**

Equation (3) is rewritten as

\[ \frac{x-y}{u-kz} = \frac{u+pz}{19(z+v)} = \frac{p}{q} \] (6)

which is equivalent to the pair of equations

\[ u = p^2 - 19q^2 \]
\[ v = 19p^2 - 2pqq^2 \] (7)

The method of cross multiplication gives (7)

Therefore, the solutions of (1) are

\[ x(p,q,k) = 19q^2(k-1)p^2+38pq(k+19) \]
\[ y(p,q,k) = 19q^2(k-1)p^2+2pq(k-19) \]
\[ z(p,q,k) = 19q^2+p^2 \]

A few interesting properties observed are as follows:

1. \( 2x(p,2,1) - y(p,2,1) - t_{10,p} \equiv -2 \) (mod 10)
2. \( z(p,1,2) - y(p,1,2) - t_{12,p} + t_{6,q} \equiv -4 \) (mod 72)
3. \( x(-3,q,2) + y(-3,q,2) - t_{12,p} + t_{14,q} \equiv -220 \) (mod 232)
4. \( y(-2,q,2) - 3Obl_{q} \equiv -33 \) (mod 65)

**PATTERN III**

Again equation (3) is rewritten as

\[ \frac{u-kz}{19(z+v)} = \frac{p}{q} \] (9)

which is equivalent to the pair of equations

\[ u = p^2 - 38pq \]
\[ v = 19p^2 - 2pqq^2 \] (10)

Using cross multiplication method, we get (10)

In view of (10) the solutions of (1) can be written as

\[ x(p,q,k) = 19q^2(k-1)p^2+38pq(k+19) \]
\[ y(p,q,k) = 19q^2(k-1)p^2+2pq(k-19) \]
\[ z(p,q,k) = 19q^2+p^2 \]

A few interesting properties observed are as follows:

1. \( 2x(p,2,1) - y(p,2,1) - t_{10,p} \equiv -8 \) (mod 126)
2. \( 2y(2,q,4) + z(2,q,4) - t_{12,q} + t_{12,p} \equiv -29 \) (mod 131)
3. \( z(p,1,3) - x(p,1,3) - S_q - t_{12,p} + t_{10,q} \equiv -2 \) (mod 13)
4. \( y(1,q,4) + x(1,q,4) - 8Obl_{q} \equiv 16 \) (mod 68)
5. \( y(-2,q,2) - 3Obl_{q} \equiv 33 \) (mod 65)

**PATTERN IV**

Equation (3) can also be written as

\[ \frac{u-kz}{19(z+v)} = \frac{p}{q} \] (11)
The Corresponding solutions of (11) is the form
\[ u=q^7-19p^7 \text{ and } v=q^7-19p^7-2pqk \]
Equation (12) the solutions of (1) can be written as
\[ x(p,q,k) = q^7(k+1)-19p^7(k+1)-2pq(k+1) \]
\[ y(p,q,k) = q^7(k+1)-19p^7(k-1)+2pq(k+1) \]
\[ z(p,q,k) = q^7+19p^2 \]
A few interesting properties observed are as follows.
1. \( y(3,q,4) - z(3,q,4) \equiv -4 \pmod{136} \)
2. \( x(4,q,1) + y(4,q,1) - t_{12,q} + t_{8,q} \equiv -302 \pmod{306} \)
3. \( x(5,q,2) + z(5,q,2) - 120 \pmod{166} \)
4. \( 2x(2,q,3) + z(2,q,3) - 90 \pmod{119} \)

**PATTERN V**
Equation (3) is equivalent to
\[ u^2 + 19v^2 = (k^2 + 19)z^2 \] 
Choose \( z = p^2 + 19q^2 \) and applying factorization method, we get
\[ u + iv\sqrt{19}v( u - iv\sqrt{19}v) = (k + i\sqrt{19})(k - i\sqrt{19}) \quad [(p + i\sqrt{19}q)(p - i\sqrt{19}q)]^2 \]
Equating real and imaginary parts in (15), we get
\[ u = p^7 - 19q^7 - 38pq \]
\[ v = p^7 - 19q^7 + 2pqk \]
Therefore the solutions of (1) are
\[ x(p,q,k) = (p^7 - 19q^7)(k+1) + 2pq(k-19) \]
\[ y(p,q,k) = (p^7 - 19q^7)(k-1) - 2pq(k+19) \]
\[ z(p,q,k) = p^7 + 19q^7 \]
Thus equation (15) represents non-zero distinct integral solutions of (1) in three parameters.
A few interesting properties observed are as follows.
1. \( x(2,q,5) - y(2,q,5) + 38pq = 8 \pmod{78} \)
2. \( z(4,q,5) - x(4,q,5) + t_{302,q} + t_{136,q} = -80 \pmod{245} \)
3. \( y(5,q,3) - 2x(5,q,3) + t_{202,q} + t_{302,q} = 150 \pmod{212} \)
4. \( 4x(p+1,1) - t_{22,p} + t_{18,p} = 0 \pmod{38} \)
5. \( y(p,p+1,2p+1) - x(p,p+1,2p+1) + t_{24,p} + 24Sqp = 38 \pmod{112} \)

**Conclusion:**
To conclude, one may search for other patterns of solutions and their corresponding properties.

**Reference:**
6. GopalanMA. Note on the Diophantine equation \( x^2 + axy + by^2 = z^2 \), ActaCiencia India 2000; XXMV(2):105-106.
7. GopalanMA. Note on the Diophantine equation \( x^2 + axy + ay^2 = 3z^2 \), ActaCiencia India 2000; XXMV(3):265-266.