A STUDY OF SEROCONVERSION TIME DISTRIBUTION OF HIV TRANSMISSION WITH GIVEN ANTIGENIC THRESHOLD

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ABSTRACT

This paper focuses on a stochastic model for the seroconversion time which deals with the damage process acting on the immune system, which is linear and cumulative with minimum antigenic diversity threshold. The number of contacts is a Poisson process and inter contact time is an exponential distribution while the threshold level is also an exponential distribution with parameter $\sigma$ and initial resistance power $\theta$ of the infected partner, then the Laplace Transform of the seroconversion time distribution is derived. The mean seroconversion time of HIV and its variance are derived.

Key words: Antigenic threshold, Exponential distribution, Laplace transformation, Inter-contact period, Seroconversion

INTRODUCTION

AIDS epidemiological data are analyzed usually by assuming some parametric distribution for the HIV infected without due regards for the dynamics of the HIV epidemic and biology of HIV. Since these distributions are determined by the dynamics of the epidemic of HIV, it is important to characterize them by the stochastic models that take into account the epidemiological, biological and social factors under considerations.

In the study of HIV epidemic disease the seroconversion time is an important component of seroconversion distribution, since the time of HIV conversion is random. One would expect that the seroconversion distribution would have an important impact on the course of HIV epidemic. In identifying AIDS patients the seroconversion time also plays a major role. Only limited studies are available in the literature for the seroconversion time of HIV transmission.

$$P_0(y) = \begin{cases} \frac{1}{\sigma} e^{-\frac{1}{\sigma}(y-\theta)} & , \theta < y < \infty, \sigma > 0 \\ 0 & , \text{other wise} \end{cases}$$

Where $\theta$ is the initial resistance power of the infected partner and $\sigma$ denotes the threshold level.
Let the seroconversion time of the HIV of the individual be the represented by the random variable $T$. We obtain the Laplace transform of the seroconversion distribution of HIV by a stochastic model based on the following assumptions.

**ASSUMPTIONS.**
1) An uninfected individual has sexual contacts with a HIV infected partner and in every contact a random number of HIV are getting transmitted.
2) A sexual contact is the only source of transmission.
3) An individual is exposed to a damage process acting on his immune system and damage process is linear and cumulative.
4) Transmissions of HIV at each contact whose inter arrival times are assumed to be identically independent random variable cause damage.
5) If the total antigenic diversity exceeds threshold level $Y$ which is itself a random variable the person is recognized as seropositive.
6) The process which generates the contacts the sequences of the damage and threshold are mutually independent.
7) From the collection of large number of inter arrival time between successive contacts of a person a random samples of $k$ observation are taken.

**NOTATIONS**

$X_i$ = A random variable representing the increase in the antigenic diversity due to the HIV transmitted during $i^{th}$ contact.

$Y$ = Random variable denoting the antigenic diversity threshold, which follows exponential distribution with the initial level threshold $\theta$ and depends on thesexual contacts.

$T$ = The continuous random variable denoting the time to Seroconversion.

$P(X<Y)$ = The Probability that the damage caused in a single contact is less than threshold $Y$

$S(t) = P \{ \text{no infection in (0,t)} \}$

$P(T>t) = \sum_{k=0}^{\infty} P \{ \text{exactly k damage in (0,t) and the Seroconversion does not take place till t} \}$

$g(.)$ = The probability density function of $X_i$

$L(.)$ = Cumulative distribution function of $T$

$L^*(.)$ = Laplace Stieltjes transform of $L(.)$

$V_i(t) = \text{Probability of exactly k damages caused to the system in [0, t]}$.

Under the above assumption with linear damage process acting on the immune system and initial resistance power of the infected partner,. We have the following theorem.

**DESCRIPTION OF THE STOCHASTIC MODEL**

Let us consider a susceptible population whose mode of transmission is through heterosexual activity. Assume that at time $t = 0$, a new member tested HIV negative enters the population and makes sexual contact with member of the susceptible population. Let the sexual contacts occur at random time points which is assumed to follow Poisson distribution with the parameter $a$ which is

$$P(X = x) = \begin{cases} e^{-a}a^x / x! & , x = 0,1,2,....,a > 0 \\ 0, & \text{Otherwise} \end{cases}$$

Let $F(t)$ be the distribution function of the inter arrival time between the contacts which follows exponential distribution. The distribution function of $F(t)$ is given by

$$F(t) = 1 - e^{-at} , t \geq 0$$
Thus stochastic model based on the cumulative damage process with the assumption that the antigenic diversity threshold is a random variable and the damage process acting on the immune system is assumed to be linear. But as the immune system are assumed to be linear. But as the immune capacities of an individual vary and also have its own resistance.

Assume the antigenic diversity threshold acting on the immune system follows an exponential distribution of the form

\[ P_\theta(y) = \begin{cases} \frac{1}{\sigma} e^{-\frac{y-\theta}{\sigma}}, & 0 < y < \infty, \sigma > 0 \\ 0, \text{ otherwise} \end{cases} \]

**TIME TO SEROCONVERSION DISTRIBUTION**

**Theorem:**

If the number of contacts is a Poisson process with the parameter \( a \) and inter contact time is an exponential distribution while the threshold level is an exponential distribution with parameter \( \sigma \) and initial resistance power \( \theta \) of the infected partner, then the Laplace Transform of the seroconversion time distribution is

\[ f(t) = \begin{cases} \alpha t [1 - g^{*}(\frac{1}{\sigma})] e^{-\alpha t}, & t > 0, \sigma > 0 \\ 0, \text{ Otherwise} \end{cases} \]

**Proof:** Consider \( S(t) = P\{\text{no infection in (0,}t\}) \)

\[ = P\{T > t\} \]

\[ = \sum_{k=0}^{\infty} P\{ \text{no seroconversion before t given exactly k contact in (0,}t\}) \]

\[ = \left\{ \sum_{k=1}^{\infty} V_k(t) \right\} X \left\{ \sum_{k=1}^{\infty} X_i < Y \right\} \]

Where \( V_k(t) = \text{Probability of k contacts in (0,}t\) \)

\[ = \frac{e^{-\alpha t}(at)^k}{k!}, k = 0,1,2,3,... \]

\[ P\{ X < Y \} = \int_{0}^{\infty} G(x)f(x)dx \]

\[ = \int_{0}^{\infty} G(x) \frac{1}{\sigma} e^{-\frac{1}{\sigma}(x-\theta)} dx \]

\[ = \frac{1}{\sigma} \int_{\theta}^{\infty} G(z+\theta)e^{-\frac{1}{\sigma}z} dZ \]

\[ = \frac{1}{\sigma} G^{*} \left( \frac{1}{\sigma} \right) \]

\[ = \frac{1}{\sigma} g^{*} \left( \frac{1}{\sigma} \right) \frac{1}{\sigma} = g^{*} \left( \frac{1}{\sigma} \right) \]

Where \( g^{*} \left( \frac{1}{\sigma} \right) = \frac{1}{\sigma} \int_{0}^{\infty} e^{-\frac{1}{\sigma}(x-\theta)} ae^{-at} dt \)
\[ P\{X_1 + X_2 + \ldots + X_k < Y\} = \int_0^\infty g_X(x) \frac{1}{\sigma} e^{-\frac{1}{\sigma}x} \, dx = \left[ g^*\left(\frac{1}{\sigma}\right)\right]^k \]

\[ g^*\left(\frac{1}{\sigma}\right) \text{ is the Laplace Transform of } g(x) \]

\[ g_x(x) = \text{p.d.f of } \sum_{k=0}^\infty X_i \]

\[ S(t) = \left(\sum_{k=0}^\infty V_i(t)\right)^k \left[ g^*\left(\frac{1}{\sigma}\right)\right]^k \]

\[ = \sum_{k=0}^\infty (at)^k \frac{e^{-ak}}{k!} \left[ g^*\left(\frac{1}{\sigma}\right)\right]^k \]

\[ = e^{-at} \sum_{k=0}^\infty \frac{(at)^k g^*\left(\frac{1}{\sigma}\right)^k}{k!} = e^{-at} e^{at} g^*\left(\frac{1}{\sigma}\right) = e^{-at} \left(1 - g^*\left(\frac{1}{\sigma}\right)\right) \]

Where

\[ 1 - g^*\left(\frac{1}{\sigma}\right) = 1 - \frac{ae^{-\theta}}{\left(\frac{1}{\sigma} + a\right)} = \frac{\left(\frac{1}{\sigma} + a\right) - ae^{-\theta}}{\left(\frac{1}{\sigma} + a\right)} \]

\[ L(t) = 1 - S(t) \text{ is called Prevalence function} \]

\[ = 1 - e^{-at} \left(1 - g^*\left(\frac{1}{\sigma}\right)\right) \]

The p.d.f of the seroconversion time

\[ f(t) = \begin{cases} 
  at[1 - g^*\left(\frac{1}{\sigma}\right)] e^{-at[1 - g^*\left(\frac{1}{\sigma}\right)]}, & t > 0, \sigma > 0 \\
  0, & \text{otherwise} 
\end{cases} \]

**PERFORMANCE MEASURES**

The performance measures of seroconversion time distribution are obtained.

\[ E[T] = \int_0^\infty [1 - g^*\left(\frac{1}{\sigma}\right)] e^{-at[1 - g^*\left(\frac{1}{\sigma}\right)]} t \, dt \]

\[ E[T] = a \left[1 - g^*\left(\frac{1}{\sigma}\right)\right] \frac{1}{a^2 [1 - g^*\left(\frac{1}{\sigma}\right)]^2} = \frac{1}{a [1 - g^*\left(\frac{1}{\sigma}\right)]} = \frac{1 \left(\frac{1}{\sigma} + a\right)}{a \left[\frac{1}{\sigma} + a - ae^{-\theta}\right]} \]

\[ E[T^2] = \frac{2a [1 - g^*\left(\frac{1}{\sigma}\right)]}{a^3 [1 - g^*\left(\frac{1}{\sigma}\right)]^3} = \frac{2}{a^2 [1 - g^*\left(\frac{1}{\sigma}\right)]^2} \]
\[ V[T] = E[T^2] - E[T] = \frac{2}{a^2 \left[1 - g \left( \frac{1}{\sigma} \right) \right]^2} - \frac{1}{a \left[1 - g \left( \frac{1}{\sigma} \right) \right]} = \frac{1}{a \left[1 - g \left( \frac{1}{\sigma} \right) \right]} \]

\[ V[T] = \frac{1}{a^2} \left[ \frac{1}{\sigma} + a - ae^{\theta} \right]^2 \]

**PARTICULAR CASES**

When the initial resistance power \( \theta = 0 \), the performance measures are

\[ E[T] = \frac{1}{\sigma} + a \quad \text{and} \quad V[T] = \frac{1}{a^2} \left( \frac{1}{\sigma} \right)^2 \]

These are the mean and variance of seroconversion time of HIV transmission when the contact rate is Poisson process and the antigenic diversity threshold is exponential, which is the same as the result of Sathiyamoorthy and Kannan (2001).

**CONCLUSION**

In the study of HIV epidemic, the seroconversion time of HIV transmission is an inevitable component. Since the sexual contact is not directly observable one would expect that the spread of AIDS would have an important impact on the human life. In this study stochastic model for time to seroconversion of HIV Transmission with antigenic threshold is proposed. The mean time to seroconversion and time of variance to seroconversion are obtained. The initial threshold level of the human body is not protecting to spread of AIDS epidemic diseases. Hence preventive measures alone protect the spread of AIDS epidemic disease. Further work is needed to develop stochastic model for controlling the spread of AIDS epidemic diseases.

**Reference**

Table 4.1  Mean time to Seroconversion when $\frac{1}{\sigma} = 0.1$

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Figure 4.1 Mean time to Seroconversion when $\frac{1}{\sigma} = 0.1$

![Graph showing mean time to seroconversion](image)

Table 4.2 Variance of Seroconversion time when $\frac{1}{\sigma} = 0.1$

<table>
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Figure 4.2 Variance of Seroconversion time when $\frac{1}{\sigma} = 0.1$

![Graph showing variance of seroconversion time](image)
### Table 4.3  Mean time to Seroconversion when $\frac{1}{\sigma} = 0.05$

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### Figure 4.3  Mean time to Seroconversion when $\frac{1}{\sigma} = 0.05$

![Graph showing mean time to seroconversion for different values of $\theta$ and $a$.](image)

### Table 4.4  Variance of Seroconversion time when $\frac{1}{\sigma} = 0.05$

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### Figure 4.4  Variance of Seroconversion time when $\frac{1}{\sigma} = 0.05$

![Graph showing variance of seroconversion time for different values of $\theta$ and $a$.](image)