CONVERGENCE OF INTERPOLATORY POLYNOMIALS

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ABSTRACT

The object of this paper is to study the convergence behaviour of a special kind of mixed type (0;0,2) interpolation on on the two sets of the nodes of Laguerre polynomial in which one set consists of the nodes of $L_n^\alpha(x)$ while the other are nodes of $L_n^{k-1}(x)$ on infinite interval.

Keywords: lacunary interpolation, Pál-Type interpolation, Laguerre Polynomial, Explicit, Representation, Estimation.

MSC 2000: 41A05, 65D32

1. Introduction

The theory of Lacunary Interpolation may be traced to a paper of G.D. Bikhoff [1], who considered a set of points $(k_0, x_0), (k_1, x_1), \ldots, (k_n, x_n)$ where $k_1, k_2, \ldots, k_n$ are non-negative integers. He obtained explicitly the interpolatory polynomial $R(x)$ such that

\begin{equation}
R^{(k)}(x_i) = 0, \quad i = 1(1)n
\end{equation}

are prescribed. The formulae obtained were so complicated that one can not do much about the convergence problem with the formulae. In 1975, L. G. Pál [10] considered an interscaled set of nodes which were the zeros of some polynomial $P(x)$ and its derivative $P'(x)$ respectively. Balázs, J.[2][3] and Szili [13] have studied problems for weighted (0,2) interpolation and T.F. XIE [15], Mathur P. and Datta S. [8] and many others mathematicians [1][4][6][7][9][11][12][14] have discussed about interpolation problems when the values of the function and its consecutive derivatives are prescribed at the given set of the points. Lénárd M. [5] investigated the Pál-type interpolation problem on the nodes of Laguerre abscissas. In this paper we study a special problem of mixed type (0;0,2) interpolation on the nodes of Laguerre polynomial. We consider the problem if $\{\xi_i\}_{i=1}^n$ and $\{\xi_i^*\}_{i=1}^n$ be the two sets of interscaled nodal points

\begin{equation}
0 \leq \xi_0 < \xi_1 < \xi_1^* < \ldots < \xi_{n-1} < \xi_n < \xi_n^* < \infty
\end{equation}

on the interval $[0, \infty)$ and $R_n(x)$ be an interpolatory polynomial of minimal possible degree $3n+k$ satisfying the interpolatory conditions:

\begin{equation}
R_n(\xi_i) = g_i, \quad R_n(\xi_i^*) = g_i^*, \quad (\omega R_n')'(\xi_i^*) = g_i^{**}, \quad \text{for} \ i = 1(1)n
\end{equation}

\begin{equation}
R_n^{(j)}(\xi_0) = g_0^{(j)}, \quad j = 0, 1, \ldots, k
\end{equation}

where $g_i, g_i^*, g_i^{**}$ and $g_0^{(j)}$ are arbitrary real numbers. We have studied the existence, uniqueness and explicit representation of interpolatory polynomial $R_n(x)$ of such type of problem earlier in [11]. The object of this paper is to study the convergence behaviour of the fundamental polynomials with respect to the weight function $\rho(x) = e^{-x/2}x^{k/2}$.
2. Preliminaries

In this section we shall give some well-known results which are as follows:
As we know that the Laguerre polynomial is a constant multiple of a confluent hypergeometric function so the differential equation is given by

\[(2.1) \quad xD^2 L_n^k(x) + (1 + k - x)DL_n^k(x) + nL_n^k(x) = 0\]

\[(2.2) \quad L_n^{(k-1)'}(x) = -L_n^{(k)}(x)\]

Also using the identities

\[(2.3) \quad L_n^{(k)}(x) = L_n^{(k-1)}(x) - L_n^{(k-1)}(x)\]

\[(2.4) \quad xL_n^{(k)'}(x) = nL_n^{(k)}(x) - (n + k)L_n^{(k)}(x)\]

We can easily find a relation

\[(2.5) \quad \frac{d}{dx}[x^kL_n^k(x)] = (n + k)x^{k-1}L_n^{(k-1)}(x)\]

By the following conditions of orthogonality and normalization we define Laguerre polynomial \(L_n^{(k)}(x)\), \(for k > -1\)

\[(2.6) \quad \int_0^\infty e^{-x}x^kL_n^{(k)}(x)L_m^{(k)}(x)dx = \Gamma k + 1 \binom{n+k}{n} \delta_{nm} \quad n, m = 0,1,2, \ldots.\]

\[(2.7) \quad L_n^{(k)}(x) = \sum_{\mu=0}^n \binom{n+k}{n} \frac{(-x)\mu}{\mu!}\]

The fundamental polynomials of Lagrange interpolation are given by

\[(2.8) \quad l_j(x) = \frac{L_n^{(k)}(x)}{L_n^{(k)}(x_j)(x-x_j)} = \delta_{ij}\]

\[(2.9) \quad l_j'(x) = \frac{L_n^{(k-1)}(x)}{L_n^{(k-1)}(x_j)(x-x_j)} = \delta_{ij}\]

\[(2.10) \quad l_j''(y_j) = \begin{cases} \frac{L_n^{(k-1)'}(y_j)}{L_n^{(k-1)}(y_j)(y_j-y_j)} & i \neq j \\ -\frac{(y_j-y_j)^2}{2y_j} & i = j \end{cases}, \quad i, j = 1(1)n\]

\[(2.11) \quad l_j'''(y_j) = \begin{cases} -\frac{L_n^{(k-1)'}(y_j)}{L_n^{(k-1)}(y_j)(y_j-y_j)} [\frac{(y_j-y_j)^2}{y_i} + \frac{2}{(y_i-y_j)}] & i \neq j \\ \frac{(y_j-y_j)(y_j-y_j)}{2y_j} & i = j \end{cases}, \quad i, j = 1(1)n\]

\[(2.12) \quad l_j''(y_j) = \frac{1}{(y_j-x_j)} \left[ l_n^{(k)'}(y_j) \frac{l_n^{(k)}(x_j)}{l_n^{(k)}(x_j)(x_j-x_j)} - l_n^{(k)'}(x_j) \frac{l_n^{(k)}(y_j)}{l_n^{(k)}(y_j)(y_j-x_j)} \right], \quad j = 1(1)n\]

For the roots of \(L_n^{(k)}(x)\) we have
3. New Result

To study the convergence behaviour, we mention here the theorem of Srivastava R. and Vishwakarma G. required in the proof of theorem (3.1)

Theorem (Srivastava R. and Vishwakarma G.)[11]: For n and k fixed positive integers, let \( \{g_i\}_{i=1}^n \), \( \{g_i^*\}_{i=1}^n \), \( \{g_i^{**}\}_{i=1}^n \) and \( \{g_0^{(j)}\}_{j=0}^k \) are arbitrary real numbers then there exists a unique polynomial \( R_n(x) \) of minimal possible degree \( \leq 3n+k \) on the nodal points (1.2) satisfying the condition (1.3) and (1.4). The polynomial \( R_n(x) \) can be written in the form

\[
R_n(x) = \sum_{j=1}^{n} U_j(x)g_j + \sum_{j=1}^{n} V_j(x)g_j^* + \sum_{j=0}^{k} C_j(x)g_0^{(j)}
\]

where \( U_j(x) \), \( V_j(x) \), \( W_j(x) \) and \( C_j(x) \) are fundamental polynomials of degree \( \leq 3n+k \) given by

\[
U_j(x) = \frac{x^{k+1}f_j(x)[l_n^{(k-1)}(x)]^2}{x_j^{(k+1)}[l_n^{(k-1)}(x)]^2}
\]

\[
V_j(x) = \frac{x^{2k+1}f_j(x)[l_n^{(k)}(x)]^2}{y_j^{(k+1)}[l_n^{(k)}(y)]^2} - \frac{x^{k+1}f_j(x)[l_n^{(k)}(x)]^2}{y_j^{(k+1)}[l_n^{(k)}(y)]^2} \int_0^x \frac{I_j(t) - x f_j^{(k-1)}(t)}{t-x} dt
\]

\[
W_j(x) = \frac{e^{y^2/2} x^{k+1}f_j(x)[l_n^{(k)}(x)]^2}{2x_j^{3k+2}l_n^{(k)}(y)[l_n^{(k)}(y)]^2} \int_0^x \frac{I_j(t)}{t-x} dt
\]

\[
C_j(x) = \frac{y^{-1/2} x^{k+1}f_j(x)[l_n^{(k)}(x)]^2}{2x_j^{3k+2}l_n^{(k)}(y)[l_n^{(k)}(y)]^2} \int_0^x \frac{I_j(t)}{t-x} dt
\]
where $U_j(x), V_j(x), W_j(x)$ and $C_j(x)$ are fundamental polynomials of degree $\leq 3n+k$. $p_j(x)$ and $q_j(x)$ are polynomials of degree at most $k-j-1$.

Now we state our main theorem.

**Theorem 3.1** Let the interpolatory function $f: \mathcal{R} \to \mathcal{R}$ be continuously differentiable such that,

$$C(m) = \{ f(x): f \text{ is continuous in } [0, \infty), f(x) = O(x^m) \text{ as } x \to \infty;$$

where $m \geq 0$ is an integer, then for every $f \in C(m)$ and $k \geq 0$

\begin{equation}
R_n(x) = \sum_{j=1}^{n} b_j^{**} U_j(x) + \sum_{j=1}^{n} b_j^{**} V_j(x) + \sum_{j=1}^{n} b_j^{**} W_j(x) + \sum_{j=0}^{k} b_j^{**} C_j(x),
\end{equation}

satisfies the relation

\begin{equation}
\rho(x) |R_n(x) - f(x)| = O \left( n^{-\frac{k}{3}} \right) \omega \left( f, \frac{\log n}{\sqrt{n}} \right), \quad \text{for } 0 \leq x \leq cn^{-1}
\end{equation}

\begin{equation}
\rho(x) |R_n(x) - f(x)| = O \left( n^{-\frac{k}{3}} \right) \omega \left( f, \frac{\log n}{\sqrt{n}} \right), \quad \text{for } cn^{-1} \leq x \leq \Omega
\end{equation}

where $\omega$ is the modulus of continuity.

The proof of the theorem (3.1) has been sketched in § 5. To prove theorem (3.1) we need the estimation of the fundamental polynomials given below:

4. **Estimation of The Fundamental Polynomials**

**Lemma 4.1.** Let the fundamental polynomial $U_j(x)$, for $j = 1, 2, ..., n$ be given by (3.2), then we have

\begin{equation}
\sum_{j=1}^{n} e^{x_j/2} x_j^{-k/2} |U_j(x)| = O \left( n^{-\frac{k}{3}} \right), \quad \text{for } 0 \leq x \leq cn^{-1}
\end{equation}

\begin{equation}
0 \left( n^{-\frac{k}{3}} \right), \quad \text{for } cn^{-1} \leq x \leq \Omega
\end{equation}

where $U_j(x)$ is given in equation (3.2).

Proof : From (3.2), we have

\begin{equation}
\sum_{j=1}^{n} e^{x_j/2} x_j^{-k/2} |U_j(x)|
\end{equation}
\[ \sum_{j=1}^{n} e^{xj^2/x_j-k/2} |I_{j}(x)| \frac{|I_{n}(k-1)(x)|^2}{|I_{j}(k+1)(x)|} \]
\[ + \sum_{j=1}^{n} e^{xj^2/x_j-k/2} |I_{j}(k)(x)| \frac{|I_{n}(k-1)(x)|^4}{|I_{j}(k+1)(x)|^2} \]
\[ \left| \int_{0}^{x} t^{(k-1)'(t)-x_j} \mu_{j}^{(k-1)}(t) \frac{dt}{t-x_j} \right| \]

Let
\[ I = \int_{0}^{x} \mu_{j}^{(k-1)}(t-x_j) \mu_{j}^{(k-1)}(t) \frac{dt}{t-x_j} \]

To evaluate \( I \), let

\[ L_{n}^{(k)}(x) = a_{j,n-1} x^{n-1} + a_{j,n-2} x^{n-2} + \ldots + a_{j,o} \]

Using (2.7) and (4.5) we get

\[ a_{j,n-1} = \frac{(-1)^{n}}{n!}, \quad a_{j,n-2} = \frac{(-1)^{n}}{n!} [x_j - n(n + k)] \]

Now let,

\[ \int_{0}^{x} \mu_{j}^{(k-1)}(t-x_j) \mu_{j}^{(k-1)}(t) \frac{dt}{t-x_j} = \sum_{l=0}^{n} C_{j,l} L_{l}^{(k)}(x) \]

comparing the coefficients in (4.7) and using (2.7) we get

\[ C_{j,n} = 1 + \frac{x_j}{n}, \]

Thus by (4.3),(4.7),(4.8) and (2.16) , we get the result.

**Lemma 4.2** Let the fundamental polynomial \( V_j(x) \), for \( j = 1,2,...,n \) be given by (3.3), then we have

\[ \sum_{j=1}^{n} e^{yj/2} y_j^{-k/2} |V_j(x)| = O \left( n^{-k/2} \right), \quad \text{for} \ 0 \leq x \leq cn^{-1} \]

\[ \sum_{j=1}^{n} e^{yj/2} y_j^{-k/2} |V_j(x)| = O \left( n^{-k/2} \right), \quad \text{for} \ cn^{-1} \leq x \leq \Omega \]

where \( V_j(x) \) is given in (3.3)

Proof : From (3.3), we have
\[ |V_j(x)| \leq \frac{|x|^{k+1} [l_j'(x)]^2}{y_j(k+1)} \left| L_n^{(k)}(x) \right| \left| L_n^{(k)}(y_j) \right| + \frac{|x|^k [L_n^{(k)}(x)] [L_n^{(k-1)}(x)]}{y_j(k+1)} \left| L_n^{(k-1)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \int_0^x l'_j(t) \, dt \]

\[ + \frac{|\tilde{c}_j| |x|^k [L_n^{(k)}(x)] [L_n^{(k-1)}(x)]}{2y_j(k+2)} \left| L_n^{(k)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \int_0^x \frac{(t-x)l_j'(x) + (k-y_j)L_n^{(k-1)}(t)}{(t-y_j)^2} \, dt \]

(4.11) \[ \sum_{j=1}^n e^{y_j/2} y_j^{-k/2} |V_j(x)| \]

\[ \leq \sum_{j=1}^n \frac{e^{y_j/2} y_j^{-k/2} [l_j'(x)]^2}{y_j(k+1)} \left| L_n^{(k)}(x) \right| \left| L_n^{(k)}(y_j) \right| \int_0^x l'_j(t) \, dt \]

\[ + \sum_{j=1}^n \frac{e^{y_j/2} y_j^{-k/2} [l_j'(x)]^2}{y_j(k+1)} \left| L_n^{(k)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \int_0^x \frac{(t-x)l_j'(x) + (k-y_j)L_n^{(k-1)}(t)}{(t-y_j)^2} \, dt \]

\[ + \sum_{j=1}^n \frac{e^{y_j/2} y_j^{-k/2} [\tilde{c}_j]}{y_j(k+2)} \left| L_n^{(k)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \int_0^x \frac{(t-x)l_j'(x) + (k-y_j)L_n^{(k-1)}(t)}{(t-y_j)^2} \, dt \]

= \zeta_1 + \zeta_2 + \zeta_3

where

\[ \zeta_1 = \sum_{j=1}^n \frac{e^{y_j/2} y_j^{-k/2} [l_j'(x)]^2}{y_j(k+1)} \left| L_n^{(k)}(x) \right| \left| L_n^{(k)}(y_j) \right| \int_0^x l'_j(t) \, dt \]

\[ \zeta_2 = \sum_{j=1}^n \frac{e^{y_j/2} y_j^{-k/2} [l_j'(x)]^2}{y_j(k+1)} \left| L_n^{(k)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \int_0^x \frac{(t-x)l_j'(x) + (k-y_j)L_n^{(k-1)}(t)}{(t-y_j)^2} \, dt \]

\[ \zeta_3 = \sum_{j=1}^n \frac{e^{y_j/2} y_j^{-k/2} [\tilde{c}_j]}{y_j(k+2)} \left| L_n^{(k)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \left| L_n^{(k-1)}(y_j) \right| \int_0^x \frac{(t-x)l_j'(x) + (k-y_j)L_n^{(k-1)}(t)}{(t-y_j)^2} \, dt \]

Follow the same procedure as discussed in Lemma (4.1) and using (4.11) and (2.16) we get the result.

**Lemma 4.3** Let the fundamental polynomial \( W_j(x) \), for \( j = 1, 2, \ldots, n \) given by (3.4), then we have

(4.12) \[ \sum_{j=1}^n |W_j(x)| = 0 \left( n^{k/2} - 2 \right) \], for \( 0 \leq x \leq cn^{-1} \)

(4.13) \[ \sum_{j=1}^n |W_j(x)| = 0 \left( n^{k/2} \right) \], for \( cn^{-1} \leq x \leq \Omega \)

where \( W_j(x) \) is given in equation (3.4)

**Proof:** From (3.4), we have
Thus by using (2.16), we yield the result.

Now we state our main theorem in § 5.

5. Proof of main theorem 3.2

Since \( R_n(x) \) given by equation (3.1) is exact for all polynomial \( Q_n(x) \) of degree \( \leq 3n+k \), we have

\[
(5.1) \quad Q_n(x) = \sum_{j=1}^{n} Q_n(x_j) U_j(x) + \sum_{j=1}^{n} Q_n(y_j) V_j(x) \\
+ \sum_{j=1}^{n} \left[ \rho(x)Q_n(x) \right]_{x=y_j}W_j(x) + \sum_{j=0}^{k} \rho(x_0) C_j(x)
\]

From equation (4.2.1) and (4.4.1) we get

\[
(5.2) \quad \rho(x)|f(x) - R_n(x)| \leq \rho(x)|f(x) - Q_n(x)| + \rho(x)|Q_n(x) - R_n(x)| \\
\leq \rho(x)|f(x) - Q_n(x)| + \sum_{j=1}^{n} \rho(x)|f(x_j) - Q_n(x_j)| |U_j(x)| \\
+ \sum_{j=1}^{n} \rho(x)|f(y_j) - Q_n(y_j)| |V_j(x)| + \sum_{j=1}^{n} \left[ \rho(x)Q_n(x) \right]_{x=y_j} |W_j(x)| \\
+ \sum_{j=0}^{k} \rho(x) |f^{(j)}(x_0) - Q_n^{(j)}(x_0)| |C_j(x)|
\]

Thus (5.2) and Lemmas 4.1 ,4.2, and 4.3 completes the proof of the theorem.

References