Some New Classes of Super Geometric Mean Graphs

S.S.Sandhya, E.Ebin Raja Merly and B.Shiny

1Department of Mathematics, Sree Ayyappa College for Women, Chunkankadai – 629 003, Kanyakumari District.
2Department of Mathematics, Nesamony Memorial Christian College, Marthandam – 629 165, Kanyakumari District.
3Department of Mathematics, DMI Engineering College, Aralvaimozhi – 629 301. Kanyakumari District.

ABSTRACT: Let \( f: V(G) \rightarrow \{1,2,\ldots,p+q\} \) be an injective function. For a vertex labeling \( \text{“} f \text{”} \), the induced edge labeling \( f^*(e=uv) \) is defined by,

\[
\begin{align*}
\left[ f(u) \right] \left[ f(v) \right] & \quad \text{or} \quad \left[ \sqrt{f(u)f(v)} \right] \quad \text{then the edge labels are distinct. In this case \text{“} f \text{”} \text{ is called a \text{“} Geometric mean labeling \text{”} of } G.}
\end{align*}
\]

A graph which admits Super Geometric mean labeling is called \text{“} Super Geometric mean graph \text{”}.

We will provide a brief summary of definitions and other informations which are necessary for our present investigation.

1. INTRODUCTION

All graphs in this paper are finite, simple and undirected graph \( G=(V,E) \) with \( p \) vertices and \( q \) edges. For a detailed survey of graph labeling we refer to Gallian [1]. For all other standard terminology and notations we follow Harary [2].

The Concept of “Geometric mean labeling” has been introduced by S.Somasundaram, R.Ponraj and P.Vidhyaran in [6].

In this paper we investigate Super Geometric mean labeling behavior of \( D(T_n)\bar{K}_1, [A(D(T_n))]\bar{K}_1, [D(Q_n)]\bar{K}_1 \) and \( [A(D(Q_n))]\bar{K}_1 \).

We will provide a brief summary of definitions and other informations which are necessary for our present investigation.

Definition: 1.1
A graph \( G=(V,E) \) with \( p \) vertices and \( q \) edges is called a “Geometric mean graph” if it is possible to label the vertices \( x \in V \) with distinct labels \( f(x) \) from \( 1,2,\ldots,q+1 \) in such a way that when each edge \( e=uv \) is labeled with, \( f(e=uv) = \left[ f(u) f(v) \right] \) or \( \left[ \sqrt{f(u)f(v)} \right] \) then the edge labels are distinct. In this case “ \( f \)” is called a “Geometric mean labeling” of \( G \).

Definition: 1.2
Let \( f: V(G) \rightarrow \{1,2,\ldots,p+q\} \) be an injective function. For a vertex labeling “ \( f \)” , the induced edge labeling \( f^*(e=uv) \) is defined by, \( f^*(e) = \left[ f(u) f(v) \right] \) or \( \left[ \sqrt{f(u)f(v)} \right] \). Then “ \( f^* \)” is called a “Super Geometric mean labeling” if \( \{f(V(G))\} \cup \{f(e):e \in E(G)\} = \{1,2,\ldots,p+q\} \). A graph which admits Super Geometric mean labeling is called “Super Geometric mean graph”.

1.3
A Triangular snake \( T_n \) is obtained from a path \( u_1u_2\ldots u_n \) by joining \( u_i \) and \( u_{i+1} \) to a new vertex \( v_i \) for \( 1 \leq i \leq n-1 \). That is every edge of path is replaced by a triangle \( C_3 \).

1.4
An Alternate Triangular Snake \( A(T_n) \) is obtained from a path \( u_1u_2\ldots u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternatively) to new vertex \( v_i \). That is every alternate edge of a path is replaced by \( C_3 \).

1.5
A Double Triangular Snake \( D(T_n) \) consists of two Triangular snakes that have a common path.

1.6
An Alternate Double Triangular snake \( A(D(T_n)) \) consists of two Alternate Triangular snakes that have a common path.

1.7
A Quadrilateral snake \( Q_n \) is obtained from a path \( u_1u_2\ldots u_n \) by joining \( u_i \) and \( u_{i+1} \) to new vertices \( v_i \) and \( w_i \) respectively and then joining \( v_i \) and \( w_i \). That is every edge of a path is replaced by a cycle \( C_4 \).

1.8
An Alternate Quadrilateral snake \( A(Q_n) \) is obtained from a path \( u_1u_2\ldots u_n \) by joining \( u_i \) and \( u_{i+1} \) (alternatively) to new vertices \( v_i \) and \( w_i \). That is every alternate edge of a path is replaced by \( C_4 \).

1.9
A Double Quadrilateral snake \( D(Q_n) \) consists of two Quadrilateral snakes that have a common path.

Key words: Graph, Geometric mean graph, Super Geometric mean graph, Double Triangular snake, Alternate Double Triangular snake, Double Quadrilateral snake and Alternate Double Quadrilateral snake.
Definition: 1.10
Alternate Double Quadrilateral snake A(D(Q_n)) consists of two Alternate Quadrilateral snakes that have a common path.

Theorem 1.11: D(T_n), A(D(T_n)), D(Q_n) and A(D(Q_n)) are Mean graphs.

Theorem 1.12: D(T_n)AK_1 and D(Q_n)AK_1 are Mean graphs.

Theorem 1.13: D(T_n), A(D(T_n)), D(Q_n) and A(D(Q_n)) are Harmonic mean graphs.

Theorem 1.14: D(T_n), A(D(T_n)), D(Q_n) and A(D(Q_n)) are Geometric mean graphs.

Theorem 1.15: D(T_n), A(D(T_n)), D(Q_n) and A(D(Q_n)) are Super Geometric mean graphs.

2. MAIN RESULTS

Theorem: 2.1
D(T_n)AK_1 is a Super Geometric mean graph.

Proof:
Let u_1u_2…u_n be the path of length n.
To construct D(T_n), let v_i and w_i, 1≤i≤n-1 be two vertices which are joined with u_i and u_{i+1}.
Let x_i be the vertex attached to v_i, 1≤i≤n-1 and y_i be the vertex attached to w_i, 1≤i≤n-1.

The resulting graph is [D(T_n)]AK_1.
Let G = [D(T_n)]AK_1.
Define a function f: V(G)→{1,2,…,p+q} by,
f(u_1) = 5
f(u_i) = 12i-11, 2≤i≤n
f(w_i) = 12i-1, 1≤i≤n-1
f(v_i) = 12i-3, 1≤i≤n-1
f(y_i) = 3
f(v_1) = 12i-7, 2≤i≤n-1.
f(x_i) = 12i-9, 2≤i≤n-1.
∴ We get distinct edge labels.
Thus both vertices and edges together get distinct labels from {1,2,…,p+q}.
Hence G is a Super Geometric mean graph.

Example 2.2: Super Geometric mean labeling of D(T_5)AK_1 is displayed below.

Theorem: 2.3
[A(D(T_n))]AK_1 is a Super Geometric mean graph.

Proof:
Let u_1u_2…u_n be the path of length n.
To construct A(D(T_n)), Let v_i and w_i be two vertices which are joined with u_i and u_{i+1} (alternatively).

Figure: 1
Let \( x_i \) be the vertex attached to \( v_i \) and \( y_i \) be the vertex attached to \( w_i \).

The resulting graph is \([A(D(T_n))]\bar{A}K_1\)

Let \( G = [A(D(T_n))]\bar{A}K_1\)

Here we consider two cases.

**Case 1:** If \( A(D(T_n)) \) starts from \( u_1 \), then we need to consider two subcases.

**Subcase (1) (a):** If “n” is odd, then
Define a function \( f: V(G) \rightarrow \{1, 2, \ldots, p+q\} \) by,

- \( f(u_1) = 5 \)
- \( f(u_{2i-1}) = 14i - 13, 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1 \)
- \( f(u_2) = 14i - 1, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \)
- \( f(v_1) = 3 \)
- \( f(v_i) = 14i - 9, 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \)
- \( f(w_i) = 14i - 3, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \)
- \( f(x_1) = 1 \)
- \( f(x_i) = 14i - 11, 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \)
- \( f(y_i) = 14i - 5, 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \)

**Figure:** 2

The labeling pattern of \([A(D(T_n))]\bar{A}K_1\) is displayed below.

∴ We get distinct edge labels.

Hence “ \( f \) ” admits a Super Geometric mean labeling of \( G \).

**Subcase (1) (b):** If “n” is even, then
Define a function \( f: V(G) \rightarrow \{1, 2, \ldots, p+q\} \) by,

- \( f(u_1) = 5 \)
- \( f(u_{2i-1}) = 14i - 13, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \)
- \( f(u_2) = 14i - 1, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \)
- \( f(v_1) = 3 \)
- \( f(v_i) = 14i - 9, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \)
- \( f(w_i) = 14i - 3, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \)
- \( f(x_1) = 1 \)
- \( f(x_i) = 14i - 11, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \)
- \( f(y_i) = 14i - 5, 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \)
The labeling pattern of $[A(D(T_6))]A_{K_1}$ is shown below. From the above labeling pattern, both vertices and edges together get distinct labels from \{1,2,\ldots,p+q\}. Hence $G$ is a Super Geometric mean graph.

**Case 2:** If $A(D(T_n))$ starts from $u_2$, then we need to consider two subcases.

**Subcase (2) (a):** If “$n$” is odd, then

Define a function $f: V(G) \rightarrow \{1,2,\ldots,p+q\}$ by,

\[
f(u_{2i-1}) = 14i-9, \quad 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1
\]

\[
f(w_i) = 14i-3, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1
\]

\[
f(y_i) = 14i-5, \quad 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor + 1
\]

\[
f(x_i) = 14i-11, \quad 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor
\]

\[
f(x_1) = 10
\]

The labeling pattern of $[A(D(T_n))]A_{K_1}$ is given below. ∴ We get distinct edge labels. Hence “$f$” provides a Super Geometric mean labeling of $G$.

**Subcase (2) (b):** If “$n$” is even, then

Define a function $f: V(G) \rightarrow \{1,2,\ldots,p+q\}$ by,

\[
f(u_{2i}) = 14i-11, \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor
\]

\[
f(v_i) = 5
\]

\[
f(v_i) = 14i-1, \quad 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor
\]

\[
f(w_i) = 14i-1, \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor
\]

\[
f(y_i) = 14i-3, \quad 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor
\]

\[
f(x_i) = 10
\]

\[
f(x_i) = 14i-9, \quad 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor
\]

\[
\text{Figure: 3}
\]
The labeling pattern of \([A(D(T_n))]A_{K_1}\) is displayed below.

The edge labels are distinct.

Thus \(G\) admits a Super Geometric mean labeling.

Hence from all the above cases, we conclude that \([A(D(T_n))]A_{K_1}\) is a Super Geometric mean graph.

\[\begin{align*}
f(u_i) &= 14i-11, \quad 1 \leq i \leq \left(\frac{n}{2}\right) \\
f(v_i) &= 5 \\
f(v) &= 14i-7, \quad 2 \leq i \leq \left(\frac{n-2}{2}\right) \\
f(w_i) &= 14i-1, \quad 1 \leq i \leq \left(\frac{n-2}{2}\right) \\
f(y_i) &= 14i-3, \quad 1 \leq i \leq \left(\frac{n-2}{2}\right) \\
f(x_i) &= 10 \\
f(x) &= 14i-9, \quad 2 \leq i \leq \left(\frac{n-2}{2}\right)
\end{align*}\]
f(n_i) = 1
f(n_i) = 20i-5, 2 ≤ i ≤ n-1
f(w_i) = 12
f(w_i) = 20i-17, 2 ≤ i ≤ n-1
f(x_i) = 19
f(x_i) = 20i-3, 2 ≤ i ≤ n-1
f(t_i) = 14
f(t_i) = 20i-14, 2 ≤ i ≤ n-1
f(v_i) = 17
f(v_i) = 20i-7, 2 ≤ i ≤ n-1

Hence \( [D(Q_n)] \) is a Super Geometric mean graph.

**Example 2.5:** Super Geometric mean labeling of \([D(Q_n)] \) is shown below.

**Theorem: 2.6**
\([A(D(Q_n))] \) is a Super Geometric mean graph.

**Proof:**
Consider a path \( u_1 u_2 \ldots u_n \).

To construct \( A(D(Q_n)) \), join \( u_i \) and \( u_{i+1} \) alternatively with four new vertices \( w_i, x_i, y_i, z_i \) by the edges \( u_i w_i, w_i x_i, x_i u_{i+1}, u_{i+1} z_i, z_i y_i, y_i u_i, \) \( 1 ≤ i ≤ n-1 \).

Let \( t_i \) and \( v_i \) be two new vertices joined with \( w_i \) and \( x_i \) respectively.

Let \( m_i \) and \( n_i \) be two new vertices joined with \( y_i \) and \( z_i \) respectively.

The resulting graph is \( [A(D(Q_n))] \) is a Super Geometric mean graph.

**Subcase (1) (a):** If “n” is odd, then
Define a function \( f: V(G) \rightarrow \{1, 2, \ldots, p+q\} \) by,
\[
f(u_1) = 10 \\
f(u_{2i-1}) = 22i-21, 2 ≤ i ≤ \left( \frac{n-1}{2} \right) + 1 \\
f(u_{2i}) = 22i-1, 1 ≤ i ≤ \left( \frac{n-1}{2} \right) \\
f(v_i) = 8
\]
f(y_i) = 22i-15, 2 \leq i \leq \left(\frac{n-1}{2}\right)
f(z_i) = 3
f(z_i) = 22i-3, 2 \leq i \leq \left(\frac{n-1}{2}\right)
f(m_i) = 4
f(m_i) = 22i-11, 2 \leq i \leq \left(\frac{n-1}{2}\right)
f(n_i) = 1
f(n_i) = 22i-7, 2 \leq i \leq \left(\frac{n-1}{2}\right)
f(w_i) = 12
f(w_i) = 22i-19, 2 \leq i \leq \left(\frac{n-1}{2}\right)
f(x_i) = 19
f(x_i) = 22i-5, 2 \leq i \leq \left(\frac{n-1}{2}\right)
f(t_i) = 14
f(t_i) = 22i-16, 2 \leq i \leq \left(\frac{n-1}{2}\right)
f(v_i) = 17
f(v_i) = 22i-9, 2 \leq i \leq \left(\frac{n-1}{2}\right)

The labeling pattern of \([A(D(Q_5))]\mathbb{A}K_1\) is given below.

∴ The edge labels are distinct.

Hence “f” provides a Super Geometric mean labeling of G.

**Subcase (1) (b):** If “n” is even, then

Define a function f: V(G)→{1,2,…,p+q} by,

f(u_1) = 10
f(u_{2i-1}) = 22i-21, 2 \leq i \leq \left(\frac{n}{2}\right)
f(u_{2i}) = 22i-1, 1 \leq i \leq \left(\frac{n}{2}\right)
f(y_i) = 8
f(y_i) = 22i-15, 2 \leq i \leq \left(\frac{n}{2}\right)
f(z_i) = 3
The labeling pattern of $[A(D(Q_n))]A\bar{K}_1$ is displayed below.

From the above labeling pattern, both vertices and edges together get distinct labels from \{1,2,...,p+q\}.

Hence $G$ is a Super Geometric mean graph.

**Case 2:** If $A(D(T_n))$ starts from $u_2$, then we need to consider two subcases.

**Subcase (2) (a):** If “n” is odd, then

Define a function $f: V(G) \to \{1,2,...,p+q\}$ by,

- $f(u_1) = 12$
- $f(u_{2i-1}) = 2i-9, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(u_{2i}) = 2i-16, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(v_1) = 17$
- $f(v_{2i-1}) = 2i-11, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(v_{2i}) = 2i-7, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(w_1) = 12$
- $f(w_{2i-1}) = 2i-19, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(w_{2i}) = 2i-15, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(x_1) = 19$
- $f(x_{2i-1}) = 2i-5, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(x_{2i}) = 2i-11, 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$
- $f(t_1) = 14$
- $f(t_{2i-1}) = 2i-21, 2 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor + 1$

The diagram illustrates the structure of the graph $A(D(T_n))A\bar{K}_1$ with the labeling pattern described.
The labeling pattern of \([A(D(Q_7))] \bar{A}K_1\) is shown below.

\[f(u_i) = 10\]
\[f(u_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(u_{2i}) = 10\]
\[f(y_i) = 7\]
\[f(y_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(y_{2i}) = 22\]
\[f(z_i) = 3\]
\[f(z_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(z_{2i}) = 3\]
\[f(m_i) = 4\]
\[f(m_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(m_{2i}) = 22\]
\[f(n_i) = 1\]
\[f(n_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(n_{2i}) = 22\]
\[f(w_i) = 17\]
\[f(w_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(w_{2i}) = 22\]

\[f(x_i) = 21\]
\[f(x_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(x_{2i}) = 22\]
\[f(t_i) = 14\]
\[f(t_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(t_{2i}) = 22\]
\[f(v_i) = 19\]
\[f(v_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n-1}{2}\right)\]
\[f(v_{2i}) = 22\]

The edge labels are distinct.
Hence \(G\) admits a Super Geometric mean labeling.

**Subcase (2) (b):** If “\(n\)” is even, then
Define a function \(f:V(G)\to\{1,2,\ldots,p+q\}\) by,
\[f(u_i) = 12\]
\[f(u_{2i-1}) = 22, \quad 2 \leq i \leq \left(\frac{n}{2}\right)\]
\[f(u_{2i}) = 10\]
The labeling pattern of \([A(D(Q_n))]\) is shown below.

\[
\begin{align*}
f(u_i) &= 22i-19, \quad 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \\
f(y_i) &= 7 \\
f(y_i) &= 22i-13, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\
f(z_i) &= 3 \\
f(z_i) &= 22i-1, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\
f(m_i) &= 4 \\
f(m_i) &= 22i-9, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\
f(n_i) &= 1 \\
f(n_i) &= 22i-5, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\
f(w_i) &= 17 \\
f(w_i) &= 22i-17, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\
f(x_i) &= 21 \\
f(x_i) &= 22i-3, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\
f(t_i) &= 14 \\
f(t_i) &= 22i-14, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor \\
f(v_i) &= 19 \\
f(v_i) &= 22i-7, \quad 2 \leq i \leq \left\lfloor \frac{n-2}{2} \right\rfloor 
\end{align*}
\]

The labeling pattern of \([A(D(Q_n))]\) is shown below.

\[
\begin{align*}
f(V(G)) \cup \{f(e) : e \in E(G)\} &= \{1, 2, \ldots, p+q\}. \\
\end{align*}
\]

Hence “f” admits a Super Geometric mean labeling of G.

\[
\begin{align*}
\therefore \quad \text{From all the above cases, we conclude that} \\
[A(D(Q_n))]\text{ is a Super Geometric mean graph.}
\end{align*}
\]
REFERENCES:


