Power Mean Labeling of some Standard Graphs

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Abstract: A graph \( G = (V, E) \) is called a Power mean graph with \( p \) vertices and \( q \) edges, if it is possible to label the vertices \( x \in V \) with distinct elements \( f(x) \) from 1, 2, 3, \( \cdots \), \( q+1 \) in such way that when each edge \( e = uv \) is labeled with

\[
\frac{1}{f(e = uv) = \frac{1}{f(u) + f(v)}}
\]
or

\[
\frac{1}{f(e = uv) = \frac{1}{f(u) + f(v)}}
\]

then the edge labels are distinct. Here \( f \) is called Power mean labeling of \( G \). We investigate on Power mean graph labeling on some standard graphs.

2010 Mathematics Subject Classification : 05C38, 05C78

Keywords and Phrases: POWER MEAN GRAPH, TRIANGULAR SNAKE \( T_n \), ALTERNATE TRIANGULAR SNAKE \( A(T_n) \), QUADRILATERAL SNAKE \( Q_n \), ALTERNATE QUADRILATERAL SNAKE \( A(Q_n) \).

1 INTRODUCTION

The graphs considered here are finite and undirected graphs. Let \( G = (V, E) \) be a graph with \( p \) vertices and \( q \) edges. For a detailed survey of graph labeling one may refer to Gallian[2] and also [1]. For all other standard terminologies and notations we follow Harary[3]. To cite some labeling techniques, we record a few of them. Somasundaram and Ponraj [6, 9] introduced and studied mean labeling for some standard graphs. Santhya et al. [4] and Sandhya and Somasundaram [5] introduced and studied Harmonic mean labeling of graphs. Somasundaram et al. [7, 8] introduced the concept of Geometric mean labeling of graphs and studied their behaviour. In this paper we investigate power mean labeling for some standard graphs like Triangular Snake \( T_n \), Alternate Triangular Snake \( A(T_n) \), Quadrilateral Snake \( Q_n \) and Alternate Quadrilateral Snake \( A(Q_n) \).

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2 DEFINITION AND RESULTS

Here we state our definition of Power mean labeling.

Definition 2.1. A graph $G = (V, E)$ with $p$ vertices and $q$ edges is said to be a Power Mean Graph if it is possible to label the vertices $x \in V$ with distinct labels $f(x)$ from $1, 2, 3, ..., q + 1$ is such a way that when each edge $e = uv$ is labeled with $f(e = uv) = \left\lceil \frac{f(u) + f(v)}{2} \right\rceil$ or $f(e = uv) = \left\lfloor \frac{f(u) + f(v)}{2} \right\rfloor$ then the resulting edge labels are distinct. In this case $f$ is called Power mean labelling of $G$.

Remark 2.1. If $G$ is a Power mean labeling graph, then 1 must be a label for one of the vertices of $G$, since an edge should get label 1.

Remark 2.2. If $p > q + 1$, then the graph $G = (p, q)$ is not a Power mean graph, since it doesn’t have sufficient labels from $\{1, 2, 3, ..., q + 1\}$ for the vertices of $G$.

The following Proposition will be used in the edge labelings of some standard graphs for Power mean labeling.

Proposition 2.1. Let $a, b$ and $i$ be positive integers with $a < b$. Then

\begin{align*}
(i) \quad & a < \left(\frac{a^p b^a}{a+b} \right)_{ab} < b, \\
(ii) \quad & i < \left(\frac{i+2}{i+2} \right) < (i + 1), \\
(iii) \quad & i < \left(\frac{i+3}{i+3} \right) < (i + 2), \\
(iv) \quad & i < \left(\frac{i+4}{i+4} \right) < (i + 2), \\
(v) \quad & \left(\frac{1}{i+1} \right)_{i+1} = \left(\frac{1}{i+1} \right) < 2.
\end{align*}

Proof. (i) Since $\frac{a^p b^a}{a+b} = a^p b^a < b^a b^a = b^a b^a$, we get the inequality in Proposition 2.1. (i). That is, the Power mean of two numbers lies between the numbers $a$ and $b$. Thus we infer that if vertices $u, v$ have labels $i, i + 1$ respectively, then the edge $uv$ may be labeled $i$ or $i + 1$ for Power mean labeling.
(ii) As a proof of this inequality, we see
\[ i^{i+2} (i+2)^i < \frac{2}{i} [i(i+2)]^i, \]
\[ < \frac{2}{i} (i+1)^{2i}, \]
since \( i(i+2) < (i+1)^2, \)
\[ < (i+1)^{2(i+1)^{2i}}, \]
\[ = (i+1)^{2i+2}. \]

This leads to \([i^{i+2} (i+2)^i \frac{}{2i+2}] < i+1. \) Therefore, if \( u, v \) have labels \( i, i+2 \) respectively, then the edge \( uv \) may be labeled \( i \) and \( i+1. \)

(iii) Next we have
\[ i^{i+3} (i+3)^i = \frac{3}{i} [i(i+3)]^i, \]
\[ < \frac{3}{i} (i+2)^{2i}, \text{ since } i(i+3) < (i+2)^2, \]
\[ < (i+2)^{3(i+2)^{2i}}, \]
\[ = (i+2)^{2i+3}. \]

This leads to \([i^{i+3} (i+3)^i \frac{}{2i+3}] < (i+2). \) Hence, if \( u, v \) have labels \( i, i+3 \) respectively, then the edge \( uv \) may be labeled \( i+1 \) without ambiguity.

(iv) Now
\[ i^{i+4} (i+4)^i = \frac{4}{i} [i(i+4)]^i, \]
\[ < \frac{4}{i} (i+2)^{2i}, \text{ since } i(i+4) < (i+2)^2, \]
\[ < (i+2)^{4(i+2)^{2i}}, \]
\[ = (i+2)^{2i+4}. \]

Therefore
\[ [i^{i+4} (i+4)^i \frac{}{2i+4}] < i+2. \]

Hence if \( u, v \) have labels \( i, i+4 \) respectively, then the edge \( uv \) may be labeled \( i+1. \)

(v) Now
\[ 2^{i+1} = (i+1)^{i+1}, \]
\[ = 1 + \frac{(i+1)}{C_1} + \cdots + \frac{(i+1)}{C_i}, \]
\[ \geq 1 + 1 + \cdots + (i+2) \text{ terms,} \]
\[ \geq i + 2 > i. \]
Therefore \((1_i 1_{i+1})_{i=1}^{i} = i_{i+1} < 2\) Thus we observe that if \(u, v\) are labeled \(1, i\) respectively, then the edge \(uv\) may be labeled 1 or 2.

### 2.1 Power Mean labeling for Triangular Snake \(T_n\)

**Triangular Snake:** A triangular Snake is a graph in which every edge of a path is replaced by \(C_3\). It is denoted by \(T_n\).

**Theorem 2.1.** Any triangular Snake \(T_n\) is a Power mean graph.

**Proof.** Let \(T_n\) be a triangular Snake with \(2n - 1\) vertices and \(3n - 3\) edges.

Define a function \(f : V(G) \rightarrow \{1, 2, 3, \ldots, q + 1 = 3n - 2\}\) as

(i) \(f(v_1) = 1,\)

(ii) \(f(v_i) = 3i - 2 \ ; \ 2 \leq i \leq n,\)

(iii) \(f(w_i) = 3i \ ; \ 1 \leq i \leq n - 1.\)

By Proposition 2.1, \((i), (ii)\) and \((iii)\) the edge are labeled

(i) \(E(v_i v_{i+1}) = 3i - 1 \ ; \ 1 \leq i \leq n - 1,\)

(ii) \(E(v_i w_i) = 3i - 2 \ ; \ 1 \leq i \leq n - 1,\)

(iii) \(E(v_{i+1} w_i) = 3i \ ; \ 1 \leq i \leq n - 1.\)

As the edges are distinct, the graph \(T_n\) is a Power mean labeled graph.

**Example 2.1.** Illustrative example for \(T_6\) in Figure 2.1.

![Figure 2.1: \(T_6\)](image)

### 2.2 Power Mean labeling for Alternate Triangular Snake \(A(T_n)\)

**Alternate Triangular Snake:** An alternate Triangular Snake \(A(T_n)\) is obtained from a path \(u_1, u_2, u_3, \ldots, u_n\) by joining \(u_i\) and \(u_{i+1}\) (alternatively) to a new vertex \(w_i\). That is, every alternate edge of a path is replaced by \(C_3\).
Theorem 2.2. Alternate Triangular Snake is a Power mean graph.

Proof. Let $A(T_n)$ be an Alternate Triangular Snake. The following two cases are to be discussed.

Case (i) : If the triangle starts from $v_1$, define a function $f : V(A(T_n)) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ as

\begin{align*}
(i) \quad & f(v_i) = 2i - 1 ; \quad 1 \leq i \leq n, \\
(ii) \quad & f(u_i) = 2i ; \quad i = 1, 3, 5, 7, \ldots, n - 1.
\end{align*}

By Proposition 2.1. (i), (ii) and (iii), the edge labels are labeled

\begin{align*}
(i) \quad & E(v_iv_{i+1}) = 2i ; \quad 1 \leq i \leq n - 1, \\
(ii) \quad & E(u_iv_i) = 2i - 1 ; \quad i = 1, 3, 5, \ldots, n - 1, \\
(iii) \quad & E(u_iv_{i+1}) = 2i + 1 ; \quad i = 1, 3, 5, \ldots, n - 1.
\end{align*}

Case (ii) : If the triangle starts from $v_2$, define a function $f : V(A(T_n)) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ as

\begin{align*}
(i) \quad & f(v_1) = 1, \\
(ii) \quad & f(v_2) = 2, \\
(iii) \quad & f(v_i) = 2i - 2 ; \quad i = 3 \leq i \leq n, \\
(iv) \quad & f(u_i) = 2i - 1 ; \quad i = 2, 4, 6, \ldots, n - 2.
\end{align*}

By Proposition 2.1. (i), (ii) and (iii), the edges are labeled

\begin{align*}
(i) \quad & E(v_i v_{i+1}) = 2i - 1 ; \quad 1 \leq i \leq n - 1, \\
(ii) \quad & E(u_i v_i) = 2i - 2 ; \quad i = 2, 4, 6, \ldots, n - 2, \\
(iii) \quad & E(u_i v_{i+1}) = 2i ; \quad i = 2, 4, 6, \ldots, n - 2.
\end{align*}

As the edges are different, any alternate triangular snake $A(T_n)$ is a Power mean labeled graph in both cases.

Example 2.2. Illustrative example for Case (i): $A(T_6)$. in Figure 2.2.
Example 2.3. *Illustrative example for Case (ii):* $A(T_6)$ in Figure 2.3.

\[
\begin{align*}
&u_2 = 3 \quad u_4 = 7 \\
&v_1 = 1 \quad v_2 = 2 \quad v_3 = 4 \quad v_4 = 6 \quad v_5 = 8 \quad v_6 = 10
\end{align*}
\]

Figure 2.3: Case (ii): $A(T_6)$

2.3 *Power Mean labeling for Quadrilateral snake* $Q_n$

**Quadrilateral snake:** A Quadrilateral snake $Q_n$ is obtained from a path $u_1, u_2, u_3, ..., u_n$ by joining $u_i$ and $u_{i+1}$ to $v_i$ and $w_i$ respectively and joining $v_i$ and $w_i$ that is every edge of a path is replaced by a cycle $C_4$.

**Theorem 2.3.** Any Quadrilateral snake $Q_n$ is a Power mean graph.

**Proof.** Let $Q_n$ be the Quadrilateral snake with $3n-2$ vertices and $4n-4$ edges (where $n$ is the number of vertices of the path). Define a function

\[
f : V(G) \rightarrow \{1, 2, 3, ..., q + 1\}
\]

as

(i) \quad f(u_i) = 4i - 3 \quad ; \quad 1 \leq i \leq n,

(ii) \quad f(v_i) = 4i - 2 \quad ; \quad 1 \leq i \leq n - 1,

(iii) \quad f(w_i) = 4i - 1 \quad ; \quad 1 \leq i \leq n - 1.

By Proposition 2.1, (i), (ii) and (iii), the edges are labeled

(i) \quad E(u_i u_{i+1}) = 4i - 2 \quad ; \quad 1 \leq i \leq n - 1,

(ii) \quad E(u_i v_i) = 4i - 3 \quad ; \quad 1 \leq i \leq n - 1,

(iii) \quad E(v_i w_i) = 4i - 1 \quad ; \quad 1 \leq i \leq n - 1,

(iv) \quad E(w_i u_{i+1}) = 4i \quad ; \quad 1 \leq i \leq n - 1.

As the edges are distinct, the Quadrilateral snake $Q_n$ is a Power mean labeled graph.
Example 2.4. Illustrative example for $Q_5$ in Figure 2.4

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{figure2_4.png}
\caption{$Q_5$}
\end{figure}

2.4 Power Mean labeling for Alternate Quadrilateral snake $A(Q_n)$

Alternate Quadrilateral snake: An alternate Quadrilateral snake $A(Q_n)$ is obtained from a path $u_1, u_2, u_3, \ldots, u_n$ by joining $u_i u_{i+1}$ (alternatively) to new vertices $v_i v_{i+1}$ respectively and then joining $v_i$ and $v_{i+1}$

ie. Every alternate edge of a path is replaced by a cycle $C_4$.

Theorem 2.4. Any Alternate Quadrilateral Snake is a Power mean graph.

Proof. Let $A(Q_n)$ be the Alternate Quadrilateral Snake.

The following two cases are to be obtained.

Case (i): Let the Quadrilateral start from $v_1$.

Define a function $f : V(A(Q_n)) \rightarrow \{1, 2, 3, \ldots, q + 1\}$ as

(i) $f(v_1) = 1,$

(ii) $f(v_2) = 4,$

(iii) $f(v_i) = f(v_{i-2}) + 5$ ; $i = 3, 4, 5, \ldots, n,$

(iv) $f(u_i) = f(v_i) - 1$ ; $i = 1, 3, 5, \ldots, n - 1,$

(v) $f(u_i) = f(v_i) - 1$ ; $i = 2, 4, 6, \ldots, n.$

By Proposition 2.1, (i), (ii) and (iii), the edges are labeled

(i) $f(v_1 v_2) = 2,$

(ii) $f(v_2 v_3) = 5,$

(iii) $f(v_i v_{i+1}) = f(v_{i-2} v_{i-1}) + 5$ ; $i = 3, 4, 5, \ldots, n - 1.$
Example 2.5. Illustrative example for be the Alternate Quadrilateral Snake $A(Q_n)$ starting from $v_1$ in Figure 2.5

![Diagram of A(Q_n) starting from v1](image)

Figure 2.5: $A(Q_n)$ starts from $v_1$

Case (ii) : Let the Quadrilateral start from $v_2$.

Define a function $f : V(A(Q_n)) \rightarrow \{1, 2, 3, \ldots, q+1\}$ as

(i) $f(v_1) = 1,$

(ii) $f(v_2) = 2,$

(iii) $f(v_3) = 5,$

(iv) $f(v_i) = f(v_{i-2}) + 5 ; \; i = 4 \leq i \leq n,$

(v) $f(u_{i-1}) = f(v_i) + 1 ; \; i = 2, 4, 6, \ldots, n - 2,$

(vi) $f(u_{i-1}) = f(v_i) - 1 ; \; i = 3, 5, 7, \ldots, n - 1.$

By Proposition 2.1., (i), (ii) and (iii), the edges are labeled

(i) $f(v_1v_2) = 1,$

(ii) $f(v_2v_3) = 4,$

(iii) $f(v_i v_{i+1}) = f(v_{i-2}v_{i-1}) + 5 ; \; 3 \leq i \leq n - 1,$

(iv) $f(v_i u_{i-1}) = f(v_i) ; \; 2 \leq i \leq n - 1,$

(v) $f(u_i u_{i+1}) = f(u_{i-1}) ; \; i = 1, 3, 5, \ldots, n - 3.$

As the edges labels are distinct, any alternate Quadrilateral Snake is a Power mean labeled graph in both cases.
Example 2.6. **Illustrative example for be the Alternate Quadrilateral Snake \(A(Q_n)\) starting from \(v_2\) in Figure 2.6**

![Diagram of \(A(Q_n)\) starting from \(v_2\)]

Figure 2.6: \(A(Q_n)\) starts from \(v_2\)

3 **CONCLUSION**

In this paper we have proved that Triangular Snake \(T_n\), Alternate Triangular Snake \(A(T_n)\), Quadrilateral Snake \(Q_n\) and Alternate Quadrilateral Snake \(A(Q_n)\) graphs are amenable for Power Mean labeling and provided illustrative examples to support our investigation.

**References**


