A NOTE ON GENERALIZED LEFT SEPARATED SPACES

ABSTRACT

The basic properties of $\mathcal{D}$-spaces are discussed and a Generalized left separated space is introduced which is a $\mathcal{D}$-space. It is also proved that every finite power of the Sorgenfrey line equipped with the topology generated by the basis $\{(a,b) / a,b \in \mathbb{R}\}$ is a GLS-Space.

INTRODUCTION

In 1979 E.K. Van Douwen and W.F. Pfeffer [1] first introduced the concept of $\mathcal{D}$-spaces. In this paper we introduce a generalized left separated space with a reflexive binary relation.

Definition 1.1 [6] A neighbourhood assignment for a topological space $(X, \tau)$ is a function $N: X \rightarrow \tau$ such that $x \in N(x)$ for each $x \in X$. $X$ is said to be a $\mathcal{D}$-space if for every neighbourhood assignment $N$, there is a closed discrete subset $\mathcal{D}$ of $X$ such that $N(x) \setminus x \in \mathcal{D}$ covers $X$.

The $\mathcal{D}$ property is a covering property [8]. Some of the basic observations immediate from the definition are compact $T_1$-spaces and $\sigma$-compact spaces are $\mathcal{D}$ [5]. Also for a $\mathcal{D}$-space the extent equals the Lindel"{o}f number of the space [7]. Questions as to which covering property implies and is implied by the $\mathcal{D}$ property are of current research interest in topology. We discuss some basic properties of $\mathcal{D}$-spaces along with a few examples of spaces which are $\mathcal{D}$.

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1.1 Properties of \( \mathcal{D} \)-spaces

**Definition 1.2.** [2] A neighbourhood assignment for a subspace \( Y \) of \( X \) is a function \( \mu \) from \( Y \) to open subsets of \( X \) having \( x \in \mu(x) \) for each \( x \in Y \).

**Property 1.3.** [4] A closed subspace of a \( \mathcal{D} \)-space is \( \mathcal{D} \).

**Definition 1.4.** [5] Extent of a space \( e(X) \) is the supremum of the cardinalities of closed discrete subsets of \( X \).

**Definition 1.5.** [7] Lindelöf degree \( L(X) \) is the least cardinal \( K \) such that every open cover of \( X \) has a subcover of cardinality \( \leq K \). For any space \( X \), \( e(X), (X) \).

**Property 1.6.** If \( X \) is a \( \mathcal{D} \)-space, \( e(X) = L(X) \).

**Proof.** Observe that if \( U \) is an open cover with no subcover as the range space of a neighbourhood assignment, since \( X \) is a \( \mathcal{D} \)-space, \( X \) has a closed discrete subset \( \mathcal{D} \) of cardinality at most \( K \). Closed discrete subsets of cardinality \( < K \) will correspond to neighbourhood assignments obtained from subcovers of cardinality \( < K \). Hence \( e(X) \equiv L(X) \).

**Proposition 1.7.** [6] If \( X \) is a \( \mathcal{D} \)-space, the \( e(Y) \equiv L(Y) \) for any closed subspace \( Y \).

**Corollary 1.8.** Countably compact \( \mathcal{D} \)-spaces are compact.

**Proof.** Let \( X \) be a countably compact \( \mathcal{D} \)-space. We need to show that every open cover has a finite subcover. From any open cover \( U \) we can construct a neighbourhood assignment, by assigning a \( U(x) \in U \) for each \( x \) with \( x \in U(x) \). Since \( X \) is a \( \mathcal{D} \)-space, this neighbourhood assignment yields a closed discrete subset \( \mathcal{D} \) of \( X \) such that \( \{U(x)/x \in \mathcal{D}\} \) covers \( X \). If we show that this \( \mathcal{D} \) is of finite cardinality then we are done. If not let there exists a closed discrete subset \( \mathcal{D} \) of \( X \) of infinite cardinality. Then choose a countably infinite subset of \( \mathcal{D} \) say \( \mathcal{D}_0 \). For each \( x \in \mathcal{D}_0 \). Let \( V(x) \) be a neighbourhood of \( X \) such that \( V(x) \cap \mathcal{D}_0 \equiv x \). \( \mathcal{D}_0 \) is closed, so \( X_0 \) is open and \( \{X_0\} \cup \{V(x)/x \in \mathcal{D}_0\} \) is then a countable open cover of \( X \) that has no finite subcover contradicting the fact that \( X \) is countably compact. Hence \( \mathcal{D} \) should be finite.

Observe that \( \omega_1 \) the space of all countable ordinals equipped with the order topology is countably compact but not compact. Hence it is not a \( \mathcal{D} \)-space. But \( \omega_1 + 1 \) is a \( \mathcal{D} \)-space. From this we can also conclude that open subspaces of a \( \mathcal{D} \)-space need not be \( \mathcal{D} \).

**Definition 1.9.** If \( \leq \) is any reflexive (not necessarily transitive) relation on a set \( E \) and \( F \subseteq E \), then \( m \) is called a \( \leq \)-minimal element of \( F \) if \( x \in F \) with \( x \leq m \), then \( x \equiv m \).

**Definition 1.10.** A space \( X \) is called a generalized left separated space (GLS-Space) if there is a reflexive binary relation \( \leq \) on \( X \), called GLS-relation, such that

1. Every non empty closed subset has a \( \leq \)-minimal element.
2. \( \{y \in X/x \leq y\} \) is open for each \( x \in X \).

**Theorem 1.11.** Every GLS-Space is a \( \mathcal{D} \)-space.
Proof. Let $X$ be a GLS-Space with GLS-relation $\preccurlyeq$. Let $\varphi$ be a neighbourhood assignment for $X$. Define a new neighbourhood assignment $\psi$ for $X$ by

$$\psi(x) = \{y \in \varphi(x) / x \preccurlyeq y\}$$

It suffices to construct a closed discrete set $\mathcal{D}$ in $X$ with $\bigcup \psi[\mathcal{D}] = X$. With transfinite recursion construct, if possible, an $x_\xi$ defined as follows:

$x_\xi$ is a $\preccurlyeq$-minimal element of $A_\xi = X \cup \{\psi(x_n) / n < \xi\}$.

We can find such an $x_\xi$ if $A_\xi \neq \emptyset$, since $A_\xi$ is closed. Let $\alpha$ be the ordinal at which the construction breaks down because $A_\alpha = \emptyset$. Let $\mathcal{D} = \{x_\xi / \psi < \alpha\}$. Then $\bigcup \psi[\mathcal{D}] = X \setminus A_\alpha = X$. To show that $\mathcal{D}$ is closed and discrete, it suffices to prove that $\psi(x) \cap \mathcal{D} = \{x\}$ for all $x \in \mathcal{D}$, since $\bigcup \psi[\mathcal{D}] = X$. Let $x_\xi \in \psi(x_\eta)$ for some $\xi, \eta < \alpha$. Then $\xi \leq \eta$ and $x_\eta \preccurlyeq x_\xi$. Both $x_\eta$ and $x_\xi$ belong to $A_\xi$. Consequently $x_\eta = x_\xi$ as $x_\xi$ is $\preccurlyeq$-minimal in $A_\xi$. Hence $\mathcal{D}$ is closed and discrete and $X$ is a $\mathcal{D}$-space.

Now let us look at some of the consequences of the above theorem.

Let $K(X)$ be the collection of non empty compact subsets of a Hausdorff space $X$. Equip $K(X)$ with the Pixely Roy topology, i.e., basic neighbourhoods about $F \in K(X)$ have the form $\{G \in K(X) / F \subseteq G \subseteq U\}$, where $U$ is an open neighbourhood of $F$ in $X$. Since ordinary inclusion is a GLS-relation, $K(X)$ is a GLS-Space and hence a $\mathcal{D}$-space.

**Lemma 1.12.** Let $\preccurlyeq$ be a reflexive and transitive binary relation on a space $X$ such that for every non empty $\preccurlyeq$-chain $K$ in $X$ there is an $m \in \overline{K}$ with $m \preccurlyeq x$ for all $x \in K$. Then each non empty closed subset of $X$ has a $\preccurlyeq$-minimal element.

**Proof.** Let $F$ be a non-empty closed subset of $X$. For every non empty $\preccurlyeq$-chain $K$ in $F$, there is an $m \in \overline{K}$ with $m \preccurlyeq x$, for all $x \in K$; so $m \in F$. Now it follows from the Kuratowski-Zorn lemma that $F$ has a minimal element.

Let $S$ denote the Sorgenfrey line equipped with the topology generated by the basis $\{(a, b) / a, b \in \mathbb{R}\}$ and $T$ be the space of all irrationals equipped with the subspace topology inherited from $S$.

**Theorem 1.13.** Every finite power of $S$ is a GLS-Space.

**Proof.** Let $H = S \setminus [0, \infty)$, half the Sorgenfrey line. Then $H$ and $S$ are homeomorphic. We shall consider $H$ instead of $S$. Let $n$ be non negative integer. As usual, the $i$th coordinate of $x \in H^n$ is $x_i, 1 \leq i \leq n$. Define a reflexive and transitive binary relation $\preccurlyeq$ on $H^n$ by

$$x \preccurlyeq y \Leftrightarrow x_i \leq y_i \text{ for all } 1 \leq i \leq n.$$ 

Then $\{y \in H^n / x \preccurlyeq y\}$ is open in $H^n$ for each $x \in H^n$. Let $K \subseteq H^n$ be a $\preccurlyeq$-chain, and define $m \in H^n$ by

$$m_i = \inf \{x_i / x \in K\}$$
Then \( m \leq x \) for each \( x \in K \), and since \( K \) is a \( \preceq \)-chain, for each \( \varepsilon > 0 \) there is an \( x \in K \) such that \( m_i \leq x_i < m_i + \varepsilon \) for \( 1 \leq i \leq n \). Consequently \( \bar{K} \). It follows from lemma(1.12) the \( \preceq \) is a GLS-relation. Hence every finite power of \( S \) is a GLS-Space.

**Theorem 1.14.** Every finite power of \( S \) is a \( \mathcal{D} \)-space.

**Proof.** The proof is a consequence of theorem(1.10) and theorem(1.12).

In fact \( S \) is hereditarily a \( \mathcal{D} \)-space. In particular \( T \) is a \( \mathcal{D} \)-space.

**Note 1.15.** The \( \mathcal{D} \)-property is not a topological property.

Consider the Sorgenfrey line \( S \) equipped with the topology generated by the basis \( ([a, b]) / a, b \in \mathbb{R} \). Both \( S \) and \( T \) are not homeomorphic since \( S \) is strongly refractible whereas \( T \) is not.

**References**


